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QUASI-STATIC STABILITY CONCEPTS AND APPLICATION TO THE LONGITUDINAL MOTION OF AN AIRCRAFT

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INTRODUCTION

This paper, an expansion and discussion of a series of lectures given by Dr. F. N. Scheubel of the Technical Institute, Darmstadt, Germany, was undertaken by the writer as a thesis project while working for the degree of Master of Science in Aeronautical Engineering at the University of Michigan, Ann Arbor, Michigan. These lectures were given at the University of Michigan during the fall of 1951 while Dr. Scheubel was visiting the University at the invitation of Dr. E. W. Conlon of the Aeronautical Engineering Department.

Dr. Scheubel presented the material in six lectures, the first three devoted to quasi-static stability concepts, the last three to their use in the solution of the equations of motion. His limited time prevented a detailed accounting of the assumptions involved as well as minute explanations of the approximations made. It is the purpose here to verify and expand on his presentation to a degree consistent with the limited scope of such a paper.

Briefly, his material covered the following. His lectures were restricted to symmetric or longitudinal motion. He developed the longitudinal equations of motion and the quasi-static stability criteria at equilibrium and at constant speed. He solved the equations using these quasi-static stability concepts for both the phugoid and short period modes for the stick-fixed case. An amplitude-phase relation for the two variables applicable to the modes was discussed. He then introduced the degree of freedom about the elevator hinge line and solved for the stick-free case. Finally the effects of an elevator impulse were discussed.

This approach to dynamic longitudinal motion differs from standard methods mainly in the quasi-static stability concepts as regards their insertion in the solutions to the equations of motion. In this respect, the method may, as illustrated, be safely applied only to conventional aircraft, i.e., a rigid body, where the number of degrees of freedom and thus the complexity of solution is restricted. The analysis of the motion assuming a phygoid and a short period oscillation serves only to illustrate the handling of the quasi-static concepts. The limitation of the method in respect to acceptable separation of the motion into these modes was

brought out by Dr. Scheubel and is discussed. The form of the solution of the complete set foregoing this separation is included in this paper.

The equations of motion as established by the standard approach of reference 1 are relegated to the flight path direction or wind axes in order that the further discussion will be based on common assumptions and approximations. Dr. Scheubel's equations as developed for a system relative to the flight path will then be compared term by term with the standard equations. From this point onward, i.e., beginning with non-dimensionalizing the equations, the presentation is as Dr. Scheubel developed it.

NOMENCLATURE

$m = W/g$	mass of the aircraft
U_1, W_1	initial velocities along x and z body axis
u', w	perturbation velocities along x and z body axis
q	angular velocity about y axis (body or wind)
X, Z, M	air forces and moments
θ, γ, α or $\Delta\theta$, $\Delta\gamma$, $\Delta\alpha$	perturbation of attitude angle, flight path angle, and angle of attack
θ_1	initial attitude angle
i_y^2	aircraft radius of gyration about y axis
$I_{yy} = mi_y^2$	moment of inertia about y axis
$V_o = w_1^2 + U_1^2$	initial flight path velocity
ΔV	velocity perturbation along flight path
$V = V_o + \Delta V$	velocity along flight path following perturbation
L, D, M	lift, drag and air moment
W	aircraft weight
T_o, D_o	initial thrust and drag
δ	elevator deflection
C_L	lift coefficient for aircraft
ρ	air density
S	wing area
C_D	aircraft drag coefficient
$u = \Delta V/V$	non-dimensional velocity perturbation, ratio of perturbation to total flight path velocity

$t_s = \frac{2W}{g \rho VS}$	adopted standard time unit
$\mu = \frac{2W}{g \quad S^{3/2}}$	mass density (See Appendix A)
$\tau = t/t_s$	non-dimensional time
M_o	initial moment about y axis
r_h	tail length, distance from aircraft c.g. to center of pressure of horizontal tail
$z = R \pm iI$	conjugate complex root of characteristic equation. R is real part, I is imaginary part
$k_y^2 = S/i_y^2$	
a_1, a_2 ...... or A_1, A_2	constant coefficients of equations
x	position of aircraft center of gravity from most forward part of aircraft in percent of wing chords
x_n	position of neutral point of aircraft from most forward part of aircraft in percent of wind chords
α_h	angle of attack of horizontal tail
C_{Lh}	lift coefficient of horizontal tail
S_h	area of horizontal tail
ω	angular frequency of mode of motion
T_p, T_r	periods of phygoid (p) and rotary (r) modes of motion
L_p	wave length of phygoid
ρ_e	distance between the hinge line and center of gravity of the elevator
m_e	mass of the elevator
i_e^2	radius of gyration of the elevator
C_h	hinge moment coefficient of elevator
c_h	mean aerodynamic chord of the horizontal tail

I_δ	elevator impulse
H_o	constant altitude reference
h	altitude change

Subscript ∞ indicates steady state values in the response discussion. Any other symbols used are either obvious or are locally defined for ease in handling equations.

II. Development of the Method.

A. The equations of longitudinal motion.

The standard development in a perfectly general way as accomplished in reference 1 result in the equations of longitudinal motion of the form,

A.1 $m(\dot{u}' + W_1 q) = X_{u'} u' + X_w w + X_q q - mg \cos \Theta_1 \theta$.

A.2 $m(\dot{w} - U_1 q) = Z_{u'} u' + Z_w w + Z_q q - mg \sin \Theta_1 \theta$.

A.3 $m\, i_y^2\, \dot{q} = I_{yy} \dot{q} = M_u\, u' + M_w w + M_q q$.

These equations are relative to body axes and contain the following assumptions and approximations:

(a) Initial symmetric steady motion is assumed.

(b) The air reactions do not depend on the rates of change of the variables, U, V and W, or their integrals.

(c) Second order and higher terms of the air reactions are neglected, i.e., only infinitesimal disturbances from an initial steady motion are treated.

(d) The aircraft has a plane of symmetey and the steady motion about which the disturbances occur is symmetrical with regard to that plane.

(e) The disturbance initially imposed on the system is unenforced and the controls are locked. This rules out an initial couple, M_o, and the variation of M with δ .

The variables are seen to be u, w and q. These are illustrated in Figure 1. The relation between the body axes and the wind axes is shown in Figure 2. The body axes are primed. Angles are measured positive counterclockwise from the horizontal reference for γ and Θ_1, from the wind line for α .

In regard to these figures it is seen that,

(a) Since q is a velocity and θ a displacement, $\theta = \int q\, dt = \Delta\alpha + \Delta\gamma$

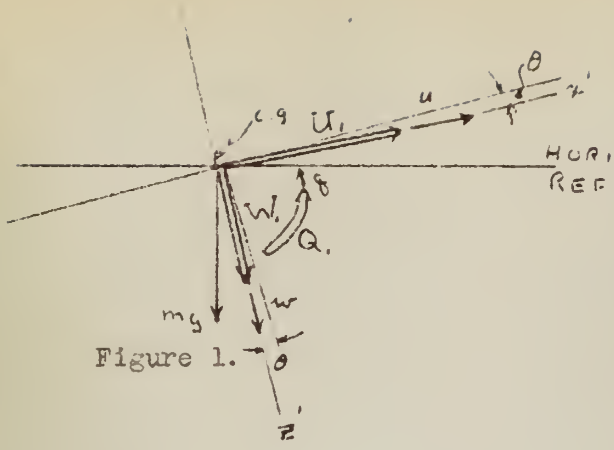


Figure 1.

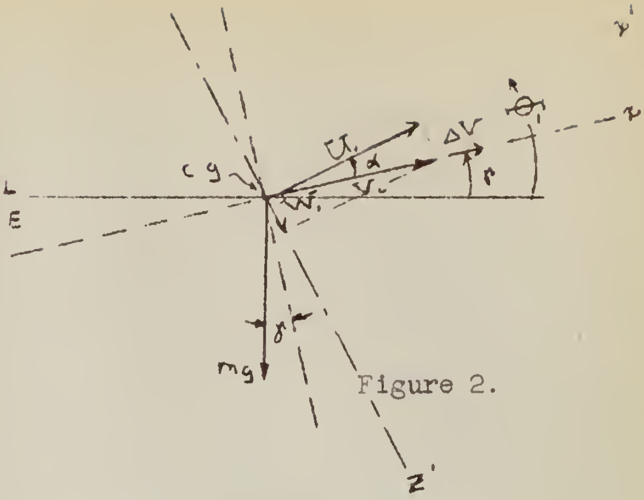


Figure 2.

(b) V_o is the steady state velocity and, $V_o^2 = W_1^2 + U_1^2$.

Then there is the definition, $V = V_o + \Delta V$

$$\text{also,} \quad \frac{dV}{dt} = \frac{d(\Delta V)}{dt}$$

(c) In essence u is now ΔV, and w does not exist.

The inertia terms of the equations of motion parallel and perpendicular to the wind direction and about the center of gravity become, neglecting the inertia force due to linear acceleration perpendicular to the wind direction,

for A.1: $\quad m \dfrac{d(\Delta V)}{dt}$

for A.2: $\quad m V \dfrac{d\gamma}{dt}$

for A.3: $\quad I_{yy} \dfrac{d^2(\alpha + \gamma)}{dt^2}$.

The weight components relative to the wind axes become,

along x: $\quad mg \sin \gamma$

along z: $\quad -mg \cos \gamma$

and these are only affected by $\Delta \gamma$.

There remains only the air reaction derivatives. The force created on the tail due to q is the largest resulting from this disturbance and from experience it is known to be small along the wind axes so X_q and Z_q are neglected.

In the standard method the left and drag contributed to the air reaction forces both along the chord and normal to it. Also velocity perturbations existed along both axes. Using wind axes, only the velocity perturbation, ΔV, now exists. Any sinking velocity along the z wind axis is merely a change in angle of attack. The air reactions along x consist only of the drag in its entirety, neglecting variations of thrust. The drag varies with both α and V. The air reactions along z consist only of the lift in its entirety and is affected by the same quantities. A pure rotary perturbation about the center of gravity changes both α and γ but only α affects the linear forces. The same applies to the air moments. The variations with α are found by wind tunnel tests and any rotation of the model about the center of gravity is considered pure α. This does not hold for the case of the effect on M of both $\Delta\dot{\alpha}$ and $\Delta\dot{\gamma}$. These two perturbations make up θ, which, when multiplied by the tail length, gives the effective sinking velocity of the horizontal tail. This sinking velocity in turn is felt as a change in angle of attack of the tail.

Thus the external forces for A.1 become,

$$\frac{\partial D}{\partial \alpha}\Delta\alpha + \frac{\partial D}{\partial V}\Delta V + \frac{\partial}{\partial \gamma}(mg \sin \gamma)\Delta\gamma$$

and equation A.1 is,

$$m\frac{d(\Delta V)}{dt} = \frac{\partial D}{\partial \alpha}\Delta\alpha + \frac{\partial D}{\partial V}\Delta V + mg \cos \gamma \, \Delta\gamma \, .$$

Equation A.2 is,

$$m V \frac{d\gamma}{dt} = \frac{\partial L}{\partial \alpha}\Delta\alpha + \frac{\partial L}{\partial V}\Delta V + mg \sin \gamma \, \Delta\gamma \, .$$

Equation A.3 becomes,

$$I_{yy}\frac{d^2(\alpha + \gamma)}{dt^2} = \frac{\partial M}{\partial \alpha}\Delta\alpha + \frac{\partial M}{\partial V}\Delta V + \frac{\partial M}{\partial \dot{\alpha}}\Delta\dot{\alpha} + \frac{\partial M}{\partial \dot{\gamma}}\Delta\dot{\gamma} \, .$$

The use of the wind axes then has the advantage of the aircraft's forward motion being along the x axis so that the lift and drag forces always

lie along these axes and resolution into normal and chord forces is not necessary. Furthermore, no resolution of the velocity is required to be made along the axis of perpendicularity which is the case if the forces are resolved along the body axes. The disadvantage is that these axes change continuously so that the moments and products of inertia change. For small disturbances from initial horizontal flight these discrepancies are considered negligible. Also the thrust does not act along the wind axes for all aircraft and, for an individual aircraft, at all times.

The continuity of this paper and the advantage of physical "feel" that is had from the development of the equations of motion from initial supposition of wind axes can be best maintained by including such a development in its entirety at this point. This is the method used by Dr. Scheubel and appears to be in general use in Germany. See Reference 2.

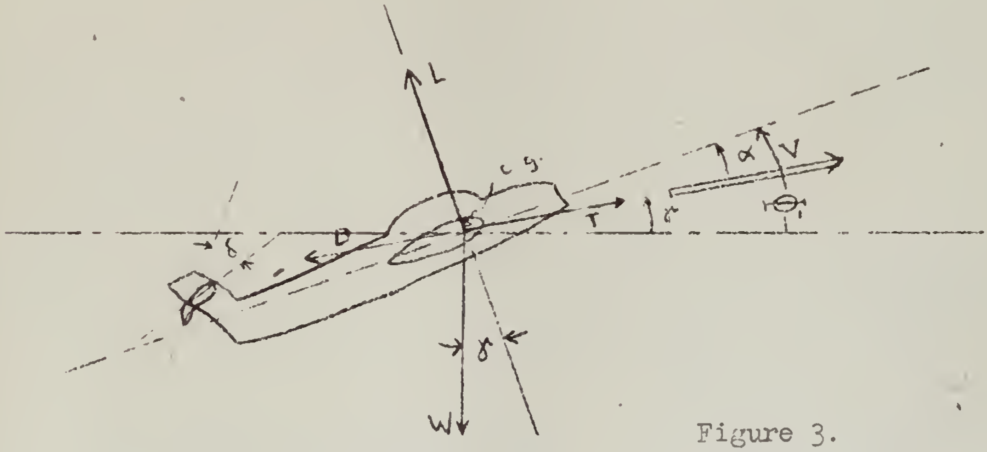


Figure 3.

With the aircraft in an attitude and with the external forces as shown in Figure 3, the external forces are equated to the time rate of change of momentum in the direction of the wind axis, i.e., along V, and one equation of equilibrium follows.

$$\frac{W}{g}\ \frac{dV}{dt} = T - D - W \sin \gamma$$

The external forces are caused to change by small changes or perturbations in the values of α , V, γ and δ . The forces in equilibrium then, following a small disturbance, are the initial values plus the increments due to the changes,

$$\frac{W}{g}\frac{dV}{dt} = T_o - D_o - W \sin \gamma_o \frac{\partial T}{\partial V}\Delta V - \frac{\partial D}{\partial V}\Delta V - \frac{\partial D}{\partial \alpha}\Delta\alpha - W \cos\gamma \Delta\gamma .$$

This is the result of a Taylor Series expansion of the quantities T, D and W as functions of α , V, γ and δ in which second order terms and above are neglected under the assumption that the changes are small. Assuming an initial condition of equilibrium, it is seen that this equation is comparable to equation A.1 with one exception. Thus all the assumptions previously enumerated hold. The exception is the thrust term. It is easily seen that thrust is not affected by changes in α , γ , or δ . The stick-fixed case is considered here also. Thus all terms involving δ are neglected.

This first equation is now manipulated into a form necessary for subsequent solution. Initial equilibrium and assuming γ small gives,

$$T_o - D_o - W \sin \gamma_o = 0 \quad \text{and,} \quad \cos\gamma = 1 \quad \text{or} \quad W = L$$

$$\text{also,} \quad L = C_L \tfrac{1}{2}\rho V^2 S \quad \text{and,} \quad D = C_D \tfrac{1}{2}\rho V^2 S \quad .$$

Multiplying terms in T by $\frac{T_o}{T_o}$ and $\frac{V}{V}$, and terms in D by $\frac{D_o}{D_o}$ and $\frac{V}{V}$ there is

$$\frac{W}{g}\frac{dV}{dt} = T_o \frac{V}{T_o}\frac{\partial T}{\partial V}\frac{\Delta V}{V} - D_o \frac{V}{D_o}\frac{\partial D}{\partial V}\frac{\Delta V}{V} - \frac{\partial D}{\partial\alpha}\Delta\alpha - L\Delta\gamma .$$

Now,

$$\frac{V}{T_o}\frac{\partial T}{\partial V} = \frac{\partial \ln T}{\partial \ln V} ; \frac{V}{D_o}\frac{\partial D}{\partial V} = \frac{\partial \ln D}{\partial \ln V} \quad \text{and letting} \quad \frac{\Delta V}{V} = u$$

there is,

$$\frac{W}{g}\frac{dV}{dt} = \frac{W}{g}\frac{d(V_o + \Delta V)}{dt} = \frac{W}{g}\frac{d(\Delta V)}{dt}$$

$$= C_D \rho \frac{V^2}{2} S \left(\frac{T_o}{D_o}\frac{\partial \ln T}{\partial \ln V} - \frac{\partial \ln D}{\partial \ln V}\right) u - \left(\frac{\partial C_D}{\partial\alpha}\Delta\alpha - C_L\Delta\gamma\right)\rho\frac{V^2}{2} S .$$

Using the standard notation, $\frac{\partial C_D}{\partial\alpha} = C_{D_\alpha}$ and dividing by $\rho \frac{V^2}{2}$ S, there is

$$\frac{W}{g}\ \frac{2}{\rho V^2 S}\ \frac{d(\Delta V)}{dt} = -C_{D_\alpha}\ \Delta\alpha - C_L \Delta\gamma - C_D \left(\frac{\partial \ln D}{\partial \ln V} - \frac{T_o}{D_o}\ \frac{\partial \ln T}{\partial \ln V}\right) u\ .$$

This equation is non-dimensionalized in the usual manner. The right side of the equation is non-dimensional as it stands. Considering the left side, multiplying numerator by V/V and the denominator by t_s/t_s,

$$\frac{W}{g}\ \frac{2}{\rho V^2 S}\ \frac{V}{t_s}\ \frac{d(\Delta V/V)}{d(t/t_s)}\ .$$

A time unit, t_s, is adopted by letting, $\frac{W}{g}\ \frac{2}{\rho V^2 S}\ \frac{V}{t_s} = 1$ then,

$t_s = \frac{2WS^{1/2}}{g\rho V S^{3/2}} = \mu\ \frac{S^{1/2}}{V} = C_L\ \frac{V}{g}$ where the mass number, μ , is given by,

$\mu = \frac{2W}{g\rho S^{3/2}} = \frac{2}{g\rho}\ (\frac{W}{S})^{3/2}\ \frac{1}{W^{1/2}}$. This mass number, or density factor, is discussed in Appendix A. Now the non-dimensional time is given by, $t/t_s = \tau$ so that, $d(\frac{t}{t_s}) = d\ \tau$ and the variable ΔV is transformed as, $\frac{\Delta V}{V} = u$.

The first equation of motion then becomes,

$$\text{A.4} \qquad \frac{du}{d\tau} = \dot{u} = -\ C_{D_\alpha}\ \Delta\alpha\ - C_L \Delta\gamma - C_D \left(\frac{\partial \ln D}{\partial \ln V} - \frac{T_o}{D_o}\ \frac{\partial \ln T}{\partial \ln V}\right) u\ ..$$

Again referring to Figure 3, the external forces perpendicular to the wind line are equated to the centrifugal force or mass times the centrifugal acceleration. $\frac{W}{g} V \frac{d\theta}{dt} = \frac{W}{g} V \frac{d\gamma}{dt} = L - W \cos \gamma$ since, $\frac{W}{g} V \frac{d\alpha}{dt} = 0$, because $V \frac{d\alpha}{dt}$ results in no change in velocity perpendicular to the flight path and the linear acceleration in this direction is neglected. If a small disturbance is introduced, and, since lift does not vary with γ and the stick-fixed case is assumed,

$$\frac{W}{g} V\ \frac{d\gamma}{dt} = L_o - W \cos\ \gamma_o + \frac{\partial L}{\partial\alpha}\Delta\alpha + \frac{\partial L}{\partial V}\Delta V + W \sin\gamma\,\Delta\gamma\ .$$

With the same assumption as before, that of initial equilibrium, this equation is seen to be the same as equation A.2 and so the same assumptions and approximations are implied. Expanding the derivatives,

$$\Delta L = \frac{\partial L}{\partial \alpha}\Delta\alpha + \frac{\partial L}{\partial V}\Delta V = \frac{\partial C_L}{\partial \alpha}\rho\frac{V^2}{2}S\Delta\alpha + \frac{\partial}{\partial V}\left(C_L\rho\frac{V^2}{2}S\right)\Delta V$$

and

$$\frac{\partial}{\partial V}\left(C_L\rho\frac{V^2}{2}S\right) = \frac{\partial C_L}{\partial V}\rho\frac{V^2}{2}S + C_L 2\rho\frac{VS}{2}\frac{V}{V} = \rho\frac{V^2}{2}S\left(\frac{C_L}{C_L}\frac{V}{V}\frac{\partial C_L}{\partial V} + \frac{2C_L}{V}\right)$$

so

$$\Delta L = \rho\frac{V^2}{2}S\left(C_{L\alpha}\Delta\alpha + C_L\left(2 + \frac{\partial \ln C_L}{\partial \ln V}\right)u\right)$$

also, $W\sin\gamma\Delta\gamma = C_L\sin\gamma\rho\frac{V^2}{2}S\Delta\gamma$, so that the entire equation becomes,

$$\frac{W}{g}\frac{2}{\rho V S}V\frac{d\gamma}{dt} = C_{L\alpha}\Delta\alpha + C_L\left(2 + \frac{\partial \ln C_L}{\partial \ln V}\right)u + C_L\sin\gamma\Delta\gamma .$$

The left side is non-dimensionalized as before. Dividing the demoninator by t_s/t_s, there is,

$$\frac{W}{g}\frac{2}{\rho V^2 S}\frac{V}{t_s}\frac{d\gamma}{d\left(\frac{t}{t_s}\right)} = \frac{d\gamma}{d\tau} = \dot\gamma .$$

So finally there is the second equation of motion,

A.5 $$\dot\gamma - C_L\left(2 + \frac{\partial \ln C_L}{\partial \ln V}\right)u - C_{L\alpha}\Delta\alpha - C_L\sin\gamma\Delta\gamma = 0 .$$

The final equation of the longitudinal motion is obtained by equating the moments to the moment of momentum about the center of gravity of the aircraft. Considering Figure 3 again, this is,

$$\frac{W}{g}i_y^2\frac{d^2(\alpha+\gamma)}{dt^2} = M_o + \Delta M$$

where, $\frac{W}{g}i_y^2$ is the moment of inertia and i_y^2 is the radius of gyration. Since M is a function of lift and drag which in turn are not affected by γ, neglecting the influence of $\dot V$, and with the stick-fixed assumption, there is,

$$\frac{W}{g}i_y^2\frac{d^2(\alpha+\gamma)}{dt^2} = I_{yy}\frac{d^2(\alpha+\gamma)}{dt^2}$$

$$= M_o + \frac{\partial M}{\partial\alpha}\Delta\alpha + \frac{\partial M}{\partial V}\Delta V + \frac{\partial M}{\partial\dot\alpha}\dot\alpha + \frac{\partial M}{\partial\dot\gamma}\Delta\dot\gamma .$$

This is seen to be the same as equation A.3 assuming initial equilibrium.

Putting the change in moment in coefficient form,

$$\Delta M = \frac{\partial C_M}{\partial \alpha} \rho \frac{V^2}{2} S^{3/2} \Delta\alpha \quad V \frac{\partial C_M}{\partial V} \rho \frac{V^2}{2} S^{3/2} \frac{\Delta V}{V} + \frac{\partial C_M}{\partial(\frac{d\theta}{dt})} \rho \frac{V^2}{2} S^{3/2} \frac{d\theta}{dt}$$

since, $\Delta\dot{\alpha} + \Delta\dot{\gamma} = \frac{d(\alpha + \gamma)}{dt} = \frac{d\theta}{dt}$.

An important point to note here is the German use of the square root of the wing area as the additional length term when converting moments to coefficient form. The reasoning is that this quantity is more easily definable for the variegated types of wings now in existence whereas the man aerodynamic chord is somewhat nebulous in definition in the literature. Another reason is the fact that the quantity $r_h/S^{1/2}$ ranges from 0.8 to 1 10 and this facilitates its approximation to unity where it appears.

Considering the last term and non-dimensionalizing the derivative by letting, $q = \frac{d\theta}{dt} \frac{S^{1/2}}{V}$,

$$\frac{\partial C_M}{\partial(\frac{d\theta}{dt})} \rho \frac{V^2}{2} S^{3/2} \frac{d\theta}{dt} = \frac{\partial C_M}{\partial q} \rho \frac{V^2}{2} S^{3/2} \frac{d\theta}{dt} \frac{S^{1/2}}{V} .$$

Then non-dimensionalizing the entire term,

$$\frac{\partial C_M}{\partial q} \rho \frac{V^2}{2} S^{3/2} \frac{d(\alpha + \gamma)}{d(t/t_s)} \frac{V}{S^{1/2}} \frac{1}{\mu} \frac{S^{1/2}}{V} = C_{M_q} \rho \frac{V^2}{2} \frac{S^{1/2}}{\mu} (\dot{\alpha} \quad \dot{\gamma}) .$$

Now non-dimensionalizing the left side of the equation, after dividing by $\rho \frac{V^2}{2} S^{3/2}$,

$$\frac{W}{g} i_y^2 \frac{2}{\rho V^2 S^{3/2}} \frac{1}{t_s^2} \frac{d^2(\alpha \quad \gamma)}{d(t/t_s)^2} = \mu i_y^2 \frac{1}{V^2} \frac{V^2}{S\mu^2} (\ddot{\alpha} + \ddot{\gamma})$$

$$= (\ddot{\alpha} \quad \ddot{\gamma}) \frac{i_y^2}{S} \frac{1}{\mu} = \Delta M .$$

This equation may be rewritten in the more concise form,

$$A.6 \; \ddot{\alpha} - C_{M_q} \frac{S}{i_y^2} \dot{\alpha} - C_{M_\alpha} \frac{S}{i_y^2} \mu \Delta\alpha \quad \ddot{\gamma} - C_{M_q} \frac{S}{i_y^2} \dot{\gamma} - \frac{\partial C_M}{\partial V} \frac{S}{i_y^2} \mu u = 0$$

There are then, the longitudinal equations of motion in non-dimensional form

A.4 $$\dot{u} + C_D \left(\frac{\partial \ln D}{\partial \ln V} - \frac{T_o}{D_o} \frac{\partial \ln T}{\partial \ln V}\right) u + C_{T_\alpha} \Delta\alpha + C_L \Delta\gamma = 0$$

A.5 $$\dot{\gamma} - C_L \left(2 + \frac{\partial \ln C_L}{\partial \ln V}\right) u - C_{L_\alpha} \Delta\alpha - C_L \sin\gamma \Delta\gamma = 0$$

A.6 $$\ddot{\alpha} - C_{m_q} \frac{S}{i_y^2} \dot{\alpha} - C_{m_\alpha} \frac{S}{i_y^2} \mu \Delta\alpha + \ddot{\gamma} - C_{m_q} \frac{S}{i_y^2} \dot{\gamma} - \frac{\partial C_m}{\partial \ln V} \frac{S}{i_y^2} \mu u = 0 \ .$$

These equations have been seen to compare term by term, except for the inclusion of variation of thrust with velocity, with these developed in a more general sense. The assumptions and approximations have been cited. The dependent variables are u, α and γ.

A more general case would be that of assuming, instead of the homogeneous set above, the equating of the quantities above to forcing functions of the elevator displacement, δ. This would require, in addition, an equation of motion involving moments about the elevator hinge line. This would be the case of stick-free stability and motion.

It is noted that a striking difference between these equations and those more familiar to most students is the form in which the stability derivatives have resulted. Since the problem is now ready for analysis of modes of motion, damping factors and associated desired results, the question arises as to why this form of the derivatives and how they are to be evaluated.

The usual procedure is the separate evaluation of each derivative by similarity to previous determinations or by wind tunnel or flight testing. It is seen that certain of the derivatives in the three equations above are in combination. For a special but usual case of the solution of the equations of motion further combinations of these derivatives appear. A discussion of both the general case and this special one of the solution follows.

B. Solution of the Equations of Motion

The most general approach to the solution of the equations of motion becomes quite involved in that the solution is carried to the determination of an equation for each variable composed of both the complimentary function and particular integral. See Reference 1. If the equations are restrictid to the stick-fixed case and the motion is assumed resulting in the absence of an applied forcing function, the equations will be homogeneous and the solution resulting will be the complimentary function only.

The discussion here will be further restricted in that the equations in terms of the variables will not be the end result This would merely give the particular motion resulting from certain initial conditions.

The characteristic equation which results from the solution of the differential equations being of the form, $x = x_o e^{z\tau}$, will be analyzed for its roots and information relative to inspection of these roots. This will show the character of the motion as regards stability, periodicity and demping.

The nature of the characteristic equation, or stability quartic, will be set down from the determinant resulting from the solution, $x = x_o e^{z\tau}$ Experience has shown that the two sets of roots that form the solution are usually of such widely separate magnitude that they may be separated. The special case mentioned is based on this fact

Since Scheubel's solutions employing the quasi-static stability concepts are based on the premise that the roots are separable, the general characteristic equation will be set down without further comment and the special case will be stressed in preparation for the solution by his quasi-static stability concepts. The errors involved in this approximation to the roots, especially as it increases with angle of attack, is discussed at length in Reference 1 It is shown that it is a good approximation for low angles of attack, i.e., 3^o, but becomes unacceptable for angles of attack near 10^o especially for neutral

stability.

1. The General Case.

Assuming a solution of the form, $x = x_o e^{z\tau}$, where z is a real or complex constant, the indicated operations on the variables are then carried out. The stability quartic results from an expansion of the following determinant that follows from the fact that the equations are consistent if the determinant of their coefficients vanishes.

Rearranging the equations by variables,

$$\begin{array}{llllll}
 & u & & & & \\
\text{A.4} & \dot{u} + C_D\left(\frac{\partial \ln D}{\partial \ln V} - \frac{T_o}{D_o}\frac{\partial \ln T}{\partial \ln V}\right)u & C_L\gamma & C_{D_\alpha}\cdot\alpha & = 0 \\
\text{A.5} & -C_L\left(2 + \frac{\partial \ln C_L}{\partial \ln V}\right)u & \dot{\gamma} - C_L \sin\gamma\cdot\gamma & -C_{L_\alpha}\cdot\alpha & = 0 \\
\text{A.6} & -\frac{\partial C_m}{\partial \ln V}\frac{S}{i_y^2}\mu u & \ddot{\gamma} - C_{m_q}\frac{S}{i_y^2}\dot{\gamma} & \ddot{\alpha} - C_{m_q}\frac{S}{i_y^2}\dot{\alpha} - C_{m_\alpha}\frac{S}{i_y^2}\mu\cdot\alpha & = 0
\end{array}$$

The assumptions are made that $C_L \sin\gamma\cdot\gamma$ is negligible since near horizontal flight is assumed, and that approximately, $T_o = D_o$.

The determinant of the coefficients is then, using $k_y^2 = S/i_y^2$

$$\begin{vmatrix}
z + C_D\left(\frac{\partial \ln D}{\partial \ln V} - \frac{\partial \ln T}{\partial \ln V}\right) & C_L & C_{D_\alpha} \\
-C_L\left(2 + \frac{\partial \ln C_L}{\partial \ln V}\right) & z & -C_{L_\alpha} \\
-\frac{\partial C_m}{\partial \ln V}k_y^2\mu & z^2 - C_{m_q}k_y^2 z & z^2 - C_{m_q}k_y^2 z - C_{m_\alpha}k_y^2\mu
\end{vmatrix}$$

The quartic becomes,

B.1 $$z^4 + a_3z^3 + a_2z^2 + a_1z + a_o = 0$$

where,

$$a_3 = C_{L_\alpha} - C_{m_q} k_y^2 + C_D \left(\frac{\partial \ln D}{\partial \ln V} - \frac{\partial \ln T}{\partial \ln V}\right)$$

$$a_2 = -\left(C_{m_\alpha} + C_{m_q} \frac{C_{L_\alpha}}{\mu}\right) k_y^2 \mu + C_D \left(\frac{\partial \ln D}{\partial \ln V} - \frac{\partial \ln T}{\partial \ln V}\right)\left(C_{L_\alpha} - C_{m_q} k_y^2\right)$$

$$+ \left(C_L - C_{D_\alpha}\right) \left[C_L\left(2 + \frac{\partial \ln C_L}{\partial \ln V}\right)\right]$$

$$a_1 = -\left(C_{m_\alpha} + C_{m_q} \frac{C_{L\alpha}}{\mu}\right) k_y^2 \mu \cdot C_D \left(\frac{\partial \ln D}{\partial \ln V} - \frac{\partial \ln T}{\partial \ln V}\right)$$

$$- \left[C_{m_\alpha} - \frac{\partial C_m}{\partial \ln V} \frac{C_{L\alpha}}{C_L\left(2 + \frac{\partial \ln C_L}{\partial \ln V}\right)}\right] k_y^2 \mu \cdot \frac{C_{D\alpha} \cdot C_L\left(2 + \frac{\partial \ln C_L}{\partial \ln V}\right)}{C_{L_\alpha}}$$

$$+ \left(C_{m_\alpha} + C_{m_q} \frac{C_{L_\alpha}}{\mu}\right) k_y^2 \mu \cdot \frac{C_{D\alpha} C_L\left(2 + \frac{\partial \ln C_L}{\partial \ln V}\right)}{C_{L_\alpha}}$$

$$- C_{m_q} k_y^2 \left[C_L^2 \left(2 + \frac{\partial \ln C_L}{\partial \ln V}\right)\right]$$

$$a_o = - C_L^2 \left(2 + \frac{\partial \ln C_L}{\partial \ln V}\right) \left[C_{m_\alpha} - \frac{\partial C_m}{\partial \ln V} \frac{C_{L\alpha}}{C_L\left(2 + \frac{\partial \ln C_L}{\partial \ln V}\right)}\right] k_y^2 \mu .$$

The usual methods for the solution of quartics may be employed to solve this equation as it stands. The constants have been arranged in the particular fashion shown for reasons to be mentioned later.

2. The Case of Distinct Sets of Roots.

Experience has shown that the usual motion in flight consists of a long period, low frequency, lightly damped motion, called the phygoid or flight path oscillation, and a short period, heavily damped motion, called the rotary oscillation. This suggests the factoring of the characteristic equation into two quadratic equations.

The length of the period of the rotary oscillation has been found to be at most of 1-5 seconds while that of the phygoid usually is of the order of 30-60 seconds for most conventional planes. This suggests that, due to the

short duration of the rotary oscillation, the velocity is essentially constant. Thus this mode is governed by equations A.5 and A.6 with the terms in u eliminated. The solution for the phygoid can neither ignore the change in velocity or angle of attack. However, consideration of equation A.6 shows that, due to the large magnitude of the factor μ discussed in Appendix A, an approximation may be made that the terms containing this item are large compared to the other terms. Solving this equation for α , the terms in α in equations A.4 and A.5 are then expressed by a term in u. The phygoid will be solved for, as regards the information mentioned previously, by consideration of the equations enclosed by the dashed lines in the equation array modified by the approximation discussed above. The rotary oscillation will be concerned by the equations enclosed by the dotted lines in the array.

a. The Phugoid.

Equation A.6 becomes, nelgecting terms not involving μ ,

$$-\frac{\partial C_m}{\partial \ln V} k_y^2/\mu\, u - C_{m_\alpha} k_y^2/\mu\, \alpha = 0$$

or,

$$\alpha = -\left(\frac{\partial C_m}{\partial \ln V}/C_{m_\alpha}\right)u \quad .$$

Equations A.4 and A.5 in coefficient form are now, assuming a solution of the form, $x = x_o e^{z\tau}$,

$$Z + C_D\left(\frac{\partial \ln D}{\partial \ln V} - \frac{\partial \ln T}{\partial \ln V}\right) - C_D\left(\frac{\frac{\partial C_m}{\partial \ln V}}{C_{m_\alpha}}\right) + C_L = 0$$

$$-C_L\left(2 + \frac{\partial \ln C_L}{\partial \ln V}\right) + C_{L_\alpha}\left(\frac{\frac{\partial C_m}{\partial \ln V}}{C_{m_\alpha}}\right) \quad Z = 0$$

which gives the quadratic,

B.2 $$Z^2 + a_1 Z + a_o = 0$$

where

$$a_1 = C_D\left(\frac{\partial \ln D}{\partial \ln V} - \frac{\partial \ln T}{\partial \ln V}\right) - C_{D_\alpha}\left(\frac{\frac{\partial C_m}{\partial \ln V}}{C_{m_\alpha}}\right)$$

and,

$$a_o = C_L^2\left(2 + \frac{\partial \ln C_L}{\partial \ln V}\right) + C_L\, C_{L_\alpha}\left(\frac{\frac{\partial C_m}{\partial \ln V}}{C_{m_\alpha}}\right)$$

$$= \frac{C_L^2\left(2 + \frac{\partial \ln C_L}{\partial \ln V}\right)}{C_{m_\alpha}}\left[C_{m_\alpha} - \frac{\partial C_m}{\partial \ln V}\quad \frac{C_{L_\alpha}}{C_L\left(2 + \frac{\partial \ln C_L}{\partial \ln V}\right)}\right]$$

The quadratic has the roots,

$$z = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_o}}{2} = R_1 \pm i\, I_1 \quad .$$

Where the damping factor, R_1, is, $R_1 = -\frac{a_1}{2}$ the frequency or period factor is, $I_1 = \sqrt{a_o - \frac{1}{4}a_1^2}$.

b. The Short Period Oscillation.

Neglecting the velocity terms, and equation A.4 in its entirety, equations A.5 and A.6 become, in coefficient form,

$$\begin{vmatrix} z & -C_{L_\alpha} \\ z^2 - C_{m_q} k_y^2 z & z^2 - C_{m_q} k_y^2 z - C_{m_\alpha} k_y^2 \mu \end{vmatrix} = 0$$

which gives, $z^3 + A_1 z^2 + A_o = 0$ B.3

where, $A_1 = (C_{L_\alpha} - C_{m_q} k_y^2)$ and, $A_o = (C_{m_\alpha} + C_{m_q} \frac{C_{L_\alpha}}{\mu})\, k_y^2 \mu$. One root is, $z = 0$, the others are, $z = R_2 + iI_2$ and the damping factor is, $R_2 = -\frac{A_1}{2}$, the frequency factor, $I_2 = \sqrt{A_o - \frac{1}{4}A_1^2}$.

Before any further evaluation of the quantities determined up to this point, the quasi-static stability criteria are developed.

C. Quasi-Static Stability

Before proceeding with the quasi-static stability development, it may be helpful if a reiteration be made of the specialization assumed leading to this development compared with methods of a more general nature as in references 1 and 4. Both of these references discuss the special case of the approximation to the roots resulting from the assumption that the solution consists of two pairs of complex roots sufficiently separated in magnitude to warrant a factoring of the characteristics quartic. This facile solution is usual in most cases but may not be sound for unconventional aircraft or even conventional aircraft where additional facts must be considered. These are, flexing of the component parts such as wings and tail surfaces of the aircraft, which would entail additional degrees of freedom, large changes in the coefficients of the equations due to rapid depletion of a large fuel supply and its effect on the derivatives. Many of these latter considerations are undergoing exhaustive study at present and do not lend themselves to any concise generalizations. From, say, a project engineer's point of view, especially in the initial stages of design, concern is given to information available from the most rapid and inexpensive data available. This is especially true at present since specifications, the military in particular, require employment of the finished design throughout large ranges in altitude, speed and weight. The assumption of the presence of two quite widely separated modes, however, is sound for most conventional aircraft and for unconventional aircraft at certain phases of their flight history. The conditions that hold for the phugoid were seen to be a slow oscillation of long period and wave length and relatively light damping. The rotary oscillation is a short, heavily damped oscillation.

In regard to standard methods, the coefficients needed to determine the constants for the final solutions of the equations of motion are obtained primarily from static wind tunnel tests. This is the rule for the values of

equations A.1 and A.2 as seen in references 1, 3 and 4. The moment coefficients are found by static stability considerations for the variation of the moment with α. See reference 5. A dynamic wind tunnel test is used to find C_{m_q}. See references 1, 4 and 6.

The coefficients as they have been developed in this paper will be evaluated in much the same manner as regards those concerned with linear forces. However, the method of evaluation of the logarithmic derivatives will be illustrated. As for the moment derivatives, the determination of C_{m_q} will not be discussed assuming that it may be found by the usual methods. The static longitudinal stability criterion, C_{m_α}, is the quanitity that the next few pages wi[l] be concerned with. This quantity Dr. Scheubel determines in a quasi-static way. Qausi-static in that it is developed from assumptions made on the dynamic motion of the aircraft. The two concepts, quasi-static stability at equilibrium and at constant speed, follow from the fact that an aircraft has two distinct phases of motion following a disturbance.

Quasi-static stability at equilibrium of forces is essentially a consideration of the equilibrium conditions reached at a time in the flight history following a disturbance at which the new velocity is obtained. This new velocity is the initial value plus the incremental increase due to the perturbation. The quasi-static stability at constant speed development holds for the relatively short period following the disturbance in which the velocity is assumed not to have reaches its final value.

The dynamic response to an elevator deflection is disected, so to speak, to give these two quasi-static concepts. The aircraft is initially in steady, nearly horizontal straight flight. Starting from this steady straight flight, we assume that the elevator angle, δ, has been changed suddenly by a certain amount, d δ, and we ask what will happen.

In the first instant, the elevator displacement gives a change in longitudinal moment and a small change in lift. This latter change is so small it is neglected. The change in moment disturbing the equilibrium gives an angular velocity about the lateral axis, and, due to this, the angle of attack is changed too. A change in the angle of attack gives rise to a change of the lift coefficient and by it a change in lift. So the equilibrium of forces perpendicular to the direction of flight is disturbed too. All these effects hold during the initial portion of the period following the disturbance, i.e., during the first few seconds. In addition, it is assumed that the speed remains essentially constant during this period.

This then is the situation maintaining for quasi-static stability at constant speed and considers the change in force perpendiicular to the flight path caused by the initial curvature of the flight path and the change in moments resulting from the disturbance.

Eventually the change in elevator causing the change in angle of attack thereby the lift coefficient results in the aircraft attaining a new steady state, that is a new point of equilibrium of forces and moments at a certain speed, V + dV, which is different from the initial one by dV. Quasi-static stability at equilibrium concerns itself with this portion of the flight history. From experience it is known that it takes an aircraft an appreciable time, normally several minutes, for the speed to adjust itself to a changed elevator displacement.

1. Quasi-static Stability at Equilibrium.

The implication of an initial steady state of motion necessitates equilibrium of forces and of moments. From Figure 4 this is seen to be,

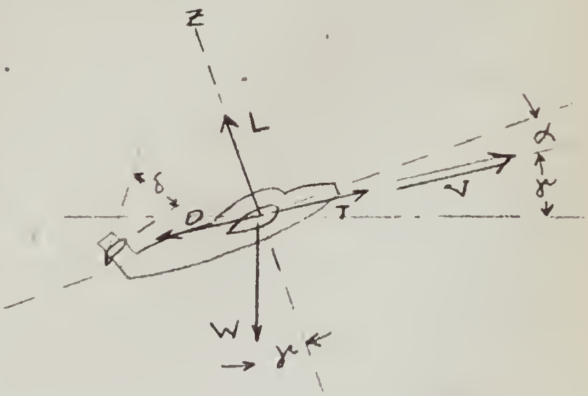


Figure 4.

$$\Sigma F_x: \quad T - D - W \sin \gamma = 0$$

$$\Sigma F_z: \quad L - W \cos \gamma = 0$$

$$\Sigma M: \quad M = 0$$

$$\Sigma M_\delta: \quad M_\delta = 0 .$$

Having started from nearly horizontal flight the flight path will remain nearly horizontal for sufficiently small changes so that the component of the weight perpendicular to the path remains almost unchanged. From the second equation, that of equilibrium of forces perpendicular to the path of flight, then,

$$d(L - W \cos \gamma) = dL = 0, \text{ since, as before } \sin \gamma = \gamma = 0.$$

now,

$$dL = dC_L \rho \frac{V^2}{2} S + 2C_L \rho \frac{V^2}{2} S \frac{dV}{V} = 0$$

$$dC_L = \frac{\partial C_L}{\partial \alpha} d\alpha + \frac{\partial C_L}{\partial V} dV \frac{VC_L}{VC_L}$$

so that,

$$\left[\frac{\partial C_L}{\partial \alpha} d\alpha + C_L \left(\frac{\partial \ln C_L}{\partial \ln V} + 2\right) \frac{dV}{V}\right] \rho \frac{V^2}{2} S = 0$$

or,

$$\frac{dV}{V} = d \ln V = - \frac{\frac{\partial C_L}{\partial \alpha}}{C_L\left(2 + \frac{\partial \ln C_L}{\partial \ln V}\right)} \cdot d\alpha$$

then finally,

C.1 $$\left.\frac{d \ln V}{d \alpha}\right|_E = - \frac{\frac{\partial C_L}{\partial \alpha}}{C_L\left(2 + \frac{\partial \ln C_L}{\partial \ln V}\right)}$$

where the subscript E indicates an equilibrium consideration with the assumptions implies above. This equation is neither a partial nor a total derivative from a mathematical point of view and caution must be exercised in handling it.

If the new state of motion is a steady one, which it must be, $dC_M = M - M_0 = 0$. The moment coefficient depends on α, V, δ, and the angular velocity, $\frac{d\theta}{dt}$. Since, however, it is assumed that the flight path has no curvature at the new steady state, and, indeed, in actuality such would be the case, the angular velocity is zero. Thus the quantity, q, does not appear.

There is then,

$$dC_M = \frac{\partial C_M}{\partial \alpha} d\alpha + \frac{\partial C_M}{\partial \delta} d\delta + \frac{\partial C_M}{\partial \ln V} d \ln V = C$$

or

$$d\alpha \left(\frac{\partial C_M}{\partial \alpha} + \frac{\partial C_M}{\partial \ln V} \frac{d \ln V}{d\alpha}\right) = -\frac{\partial C_M}{\partial \delta} d\delta$$

thus,

$$\left.\frac{d\alpha}{d\delta}\right|_E = -\frac{\frac{\partial C_M}{\partial \delta}}{\frac{\partial C_M}{\partial \alpha} + \frac{\partial C_M}{\partial \ln V} \left.\frac{d \ln V}{d\alpha}\right|_E} = -\frac{\frac{\partial C_M}{\partial \delta}}{\left.\frac{dC_M}{d\alpha}\right|_E}$$

where,

C.2 $$\left.\frac{dC_M}{d\alpha}\right|_E = \frac{\partial C_M}{\partial \alpha} + \frac{\partial C_M}{\partial \ln V} \left.\frac{d \ln V}{d\alpha}\right|_E .$$

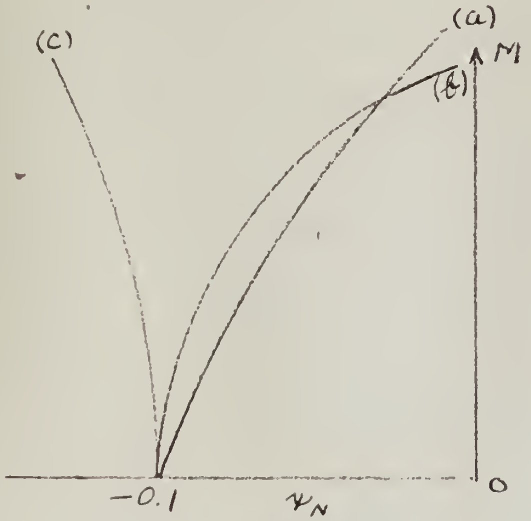
This then, is a new stability requirement that applies to that phase of the longitudinal motion described and implied from the previous discussion. This requirement certainly is sound. For an aircraft which does not have this quasi-static stability will be unstable in as much as for any disturbed state of motion, enforced by an elevator displacement, will show the tendency to move its elevator further in this direction, so increasing the deviation from its initial state.

The usual static stability criterion, $\frac{dC_M}{d\alpha}$, is considered either, $-\frac{dC_M}{dC_L} \frac{dC_L}{d\alpha}$ or, $-\frac{dC_L}{d\alpha}(x - x_N)$. The quantity, $\frac{dC_L}{d\alpha}$, in either of the above is a constant throughout the usual range of normal flight, i.e, unstalled flight. The value of $x - x_N$ can be found graphiclly from wind tunnel data of C_M vs. C_L as explained in reference 5. This is the more informative quantity since a shift in the center of gravity is more readily determined throughout the flight history.

The first term in equation C.2 is very similar to the static stability criterion above. However for comparison, equation C.2 is written in its entirety as,

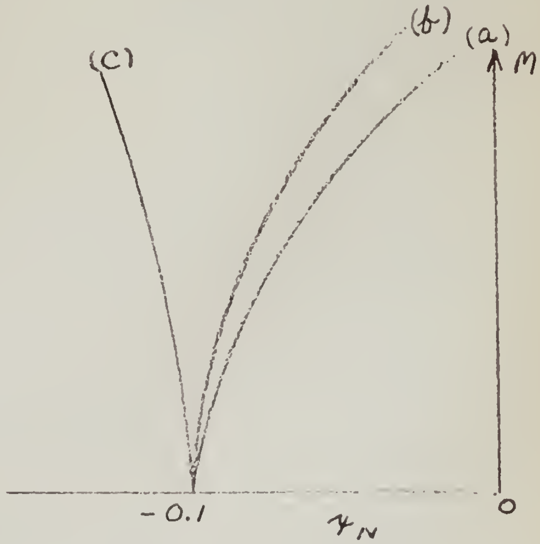
$$\frac{dC_M}{d\alpha}\bigg|_E = -\frac{dC_L}{d\alpha}\bigg|_E \left(x - x_N\big|_E\right)$$

where $x_N|_E$ is a new neutral point that takes into account variation of the neutral point with velocity, i.e., compressibility effects. First, it is noted that variation of lift coefficient with angle of attack differs between the two. This is due to the fact that the ordinary determination of the change of lift coefficient with angle of attack is accomplished in wind tunnel tests where the stream velocity is kept constant. The results are then based on a constant dynamic pressure, whereas the present development is not. The difference between the two is about 1-3o/o and is sumed negligible. The quantity $x_N|_E$ moves with respect to velocity, dynamic pressure, deflection of the fuselage, twist of the wings and horizontal stabilizer. A comparison with the movement of the usual static stability neutral point will be made only with respect to variation with velocity and dynamic pressure. Figure 5 shows the variation with velocity of three different aspects of stability at sea level. These are: (a) static stability only, (b) quasi-static stability assuming the zero lift moment coefficient, $C_{M_{L_0}}$, to be -0.02, and (c) quasi-static stability with zero lift coefficient assumed zero. Figure 6 shows the comparison of the same quantities at altitude. Here the decrease in the stability margin



H = Sea Level

Figure 5



H = 40,000 ft.

Figure 6

for quasi-static stability is much less due to the presence of the density in the dynamic pressure in the denominator of the additional term, i.e., C_L in $\left.\frac{d \ln V}{d\alpha}\right|_E$.

2. Quasi-Static Stability at Constant Speed.

Recalling the discussion of the conditions holding following the imposing of a small elevator deflection on the steady state, it was seen that in the first small interval following the change the speed does not change appreciably. Thus the quasi-static stability at equilibrium concept does not hold for this phase of the disturbed motion. It was further seen that the equilibrium of forces perpendicular to the direction of flight is disturbed too. The equation of motion for this direction shows what happens.

If a force is exerted on a body moving in a certain direction, its path will be curved. See Figure 7. If an aircraft, the motion may become rather complicated, for, due to the curvature of the flight path in a vertical plane, the component force of the weight perpendicular to the path changes too. Restriction to small disturbances close to horizontal flight allows neglecting the changes in the weight component.

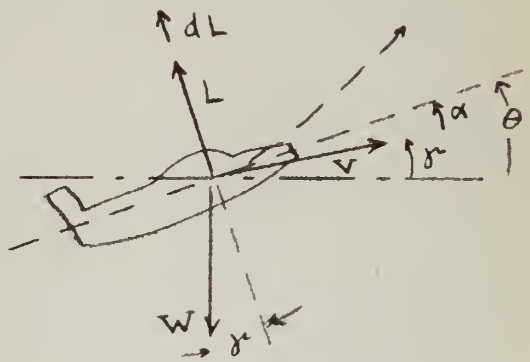


Figure 7.

For instance, a change in angle of flight path of 15° gives only 3 o/o change. The centripetal acceleration is $V\frac{d\gamma}{dt}$. The equation of motion is therefore,

$$\frac{W}{g} V \frac{d\theta}{dt} = d(L - W\cos\gamma) = dL$$

and

$$dL = \left[\frac{\partial C_L}{\partial\alpha} d\alpha + \frac{\partial C_L}{\partial(\frac{d\theta}{dt})}\frac{d\theta}{dt}\right]\rho\frac{V^2}{2}S$$

$$= \left(\frac{\partial C_L}{\partial\alpha} d\alpha + \frac{\partial C_L}{\partial q} dq\right)\rho\frac{V^2}{2}S \quad ,$$

therefore,

$$\frac{d\theta}{dt} = \left(\frac{\partial C_L}{\partial \alpha}\, d\alpha + \frac{\partial C_L}{\partial q}\, dq\right)\rho\, \frac{V^2}{2}\; \frac{S}{m}$$

and, multiplying by $\frac{S^{1/2}}{V}$ and noting that $\frac{1}{\mu} = \rho\,\frac{V^2 S^{3/2}}{2m}$, there is

$$\mu\, dq\,\left(1 - \frac{\frac{\partial C_L}{\partial q}}{\mu}\right) = \frac{\partial C_L}{\partial \alpha}\, d\alpha$$

or,

C.3 $$\left.\frac{dq}{d\alpha}\right|_V = \frac{\frac{\partial C_L}{\partial \alpha}}{1 - \frac{\frac{\partial C_L}{\partial q}}{\mu}} \cdot \frac{1}{\mu} \quad .$$

The term, $\frac{\partial C_L/\partial q}{\mu}$, may be neglected since $\frac{\partial C_L}{\partial q}$ is small compared to $\frac{\partial C_L}{\partial \alpha}$ and, also, the usually large magnitude of μ makes it negligible compared to unity. This is also the common approximation as seen by the neglecting of X_q and Z_q in references 1, 3 and 4.

Thus there is the relation, $\left.\frac{dq}{d\alpha}\right|_V = \frac{\partial C_L}{\partial \alpha}\;\frac{1}{\mu}$, resulting from a consideration of the forces perpendicular to that of flight in the first instant following a disturbance.

This curvature of the flight path or angular velocity has an influence on the equilibrium of moment about the lateral axis. So a consideration of the equilibrium of moments along this curved flight path involves the change in elevator deflection, the change in angle of attack and this angular velocity. The effect of velocity is omitted because of the assumptions of this phase of the motion. There is then,

$$dC_M = \frac{\partial C_M}{\partial \alpha}\, d\alpha + \frac{\partial C_M}{\partial q}\left.\frac{dq}{d\alpha}\right|_V d\alpha + \frac{\partial C_M}{\partial \delta}\, d\delta = 0$$

or,

$$d\alpha = -\,\frac{\frac{\partial C_M}{\partial \delta}}{\frac{\partial C_M}{\partial \alpha} + \frac{\partial C_M}{\partial q}\left.\frac{dq}{d\alpha}\right|_V}\; d\delta \quad .$$

This expression is similar to the one found for the equilibrium of a steady state of flight enforces by a certain elevator displacement. It contains the

stability term,

$$\text{C.4} \qquad \left.\frac{dC_M}{d\alpha}\right|_V = \frac{\partial C_M}{\partial \alpha} + \frac{\partial C_M}{\partial q} \left.\frac{dq}{d\alpha}\right|_V$$

called "quasi-static stability at constant speed". A comparison of this quantity with the usual static stability is seen following a further evaluation.

The angular velocity causes the angle of attack of the horizontal tail plain, α_h, to change by the amount,

$$d\alpha_h = \frac{\frac{d\theta}{dt} r_h}{V} \qquad \text{but,} \qquad \frac{d\theta}{dt} = dq \frac{V}{S^{1/2}}$$

therefore,

$$\frac{d\alpha_h}{dq} = \frac{r_h}{S^{1/2}} \; .$$

This change in tail angle of attack gives rise to an increase in lift coefficient of the horizontal tail plane, C_{L_h}, and by it a decrease in moment about the lateral axis,

$$dC_M = - dC_{L_h} \frac{S_h r_h}{S^{3/2}}$$

and, having noted its dependence on angular velocity, there is,

$$\frac{dC_M}{dq} = - \frac{\partial C_{L_h}}{\partial \alpha_h} \frac{S_h r_h}{S^{3/2}} \frac{\partial \alpha_h}{dq} = - \frac{\partial C_{L_h}}{\partial \alpha_h} S_h \left(\frac{r_h}{S}\right)^2 \; .$$

This quantity, C_{M_q}, may be determined by a dymanic wind tunnel test described in reference 1, or by use of the curved-flow technique referred to in reference 6.

The term, $\frac{\partial C_M}{\partial q} \left.\frac{dq}{d\alpha}\right|_V$, added to the static stability is always negative, since C_{M_q} is a damping derivative. Thus this is an increase in static stability and the neutral point corresponding to this quasi-static stability is behind the one for static stability by about, $\frac{K}{\mu}$, where, $K = C_{M_q} C_{L_\alpha}$.

This apparent improvement over wind tunnel static stability is seen to be larger for an aircraft of small wing loading at low altitude than for one of the high wing loading at high altitude.

The additional term added to the usual static stability neutral point looks suspiciously like the stick-fixed maneuver point, especially because of the statement of the situation and the assumptions made. This is indeed the case and can be verified by comparison with the maneuver point found in ference 4.

Concerning the concepts of quasi-static stability, it is seen that both are improvements over wind tunnel static stability. However, three facts should be borne in mind concerning their effect and their use. First, both contain the usual static stability plus an additional term Secondly, their application and use only hold for specific conditions perculiar to different phases of the flight history Lastly, for a specific aircraft they vary differently with like parameters, i.e., density, but all parameters do not effect both.

Dr. Scheubel pointed that in Germany during the last war certain of their fighters showed a marked improvement in longitudinal stability over that predicted by wind tunnel tests at medium altutudes. Since these were of the 450-500 mph class, this was explainable by the above quasi-static stability considerations. He said further, however, that pilots complained of poor stability at high speeds at high altutudes. In as much as both concepts still predict an improvement, greater for equilibrium and less for constant speed, the aeronautical engineers were greatly puzzled. Further study accounted for this by reason of flexing of the wings, the tail plane and the fuselage. The ac ountability of these phenomena in stability studies is very complex. For the longitudinal case a good approximation made be obtained from grouping their overall effect as a shifting of the neutral point. The seriousness of their effect can be seen from mention of two simples aspects. First, upon the sudden application of an elevator deflection or a change in speed as in a gust, the elasticity of present designs at high speeds causes a twisting of the wing. This causes the angle of attack to very spanwise resulting in a shifting of the center of

pressure. The same applies to the horizontal tail surface. Secondly, a shifting of neutral point vertically from flexing of the fuselage coupled with the very large drag forces at high speeds produces an appreciable destabilizing moment.

That the problem is complex analytically is easily seen from merely a superficial study of the present day literature as, for instance, reference 7. Dr. Scheubel pointed out the experimental difficulties in a description of an attempt in Germany to determine the deflection of a horizontal stabilizer with speed. This involved a complex optical system employing an arrangement of mirrors reflecting a slit light source the movement of which was recorded by a motion picture camera. In addition to this the test aircraft was required to dive along a path determined by an array of floodlights to insure perfect linearity of flight path.

D. Evaluation of the Derivatives.

Some of the quantities to be evaluated are identical to those appearing in standard references. Methods for the determination of these will be mentioned but not discussed in detail.

Since all quantities appear in the general solution, the items will be discussed in the order that they appear in the coefficients on page 17.

C_{L_α} . Static wind tunnel tests.

C_{M_q} . Dynamic wind tunnel test by the whirling arm or oscillation methods mentioned in reference 1 or by the curved-flow technique referred to in reference 6.

$\frac{\partial \ln D}{\partial \ln V}$. A plot of C_D vs. V can be obtained from static wind tunnel tests

Then,

$$D = C_D \rho \frac{V^2}{2} S$$

$$\frac{dD}{dV} = \frac{\partial C_D}{\partial V} \rho \frac{V^2}{2} S + 2C_D \rho \frac{V^2}{2} S \frac{1}{V}$$

$$= \frac{C_D}{V} \rho \frac{V^2}{2} S \left(\frac{\partial C_D}{\partial V} \frac{V}{C_D} + 2\right)$$

$$\frac{V}{D}\frac{dD}{dV} = \frac{d \ln D}{d \ln V} = \frac{\partial \ln C_D}{\partial \ln V} + 2 .$$

Thus this varies from a value of 2 at zero M up to about 3 for M = 1.0.

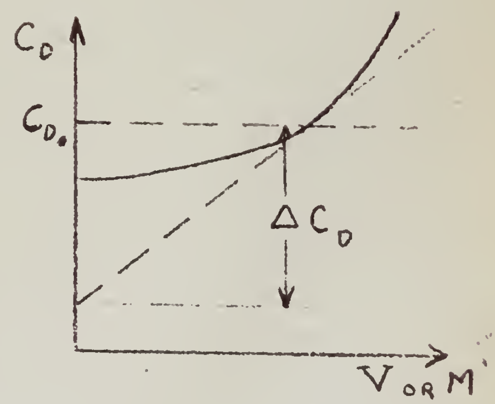


Figure 8

$\frac{\partial \ln T}{\partial \ln V}$ and $\frac{\partial \ln C_L}{\partial \ln V}$. These are considered together because they are found in much the same manner. From wind tunnel tests C_L vs. V are found. Engine manufacturers usually submit data of T vs. V for specific installations. These derivatives are then determined as in Figures 9 and 10. It is pointed out that Figure 10 is plotted for a constant angle of attack.

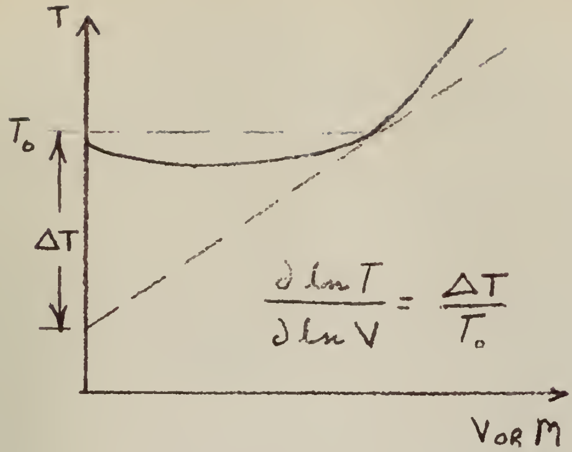


Figure 9.

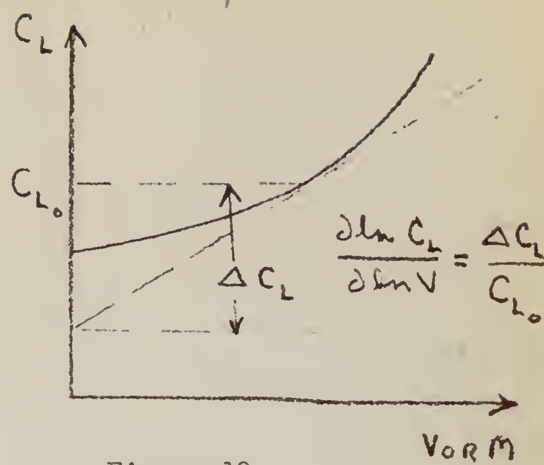


Figure 10.

$\frac{\partial C_M}{\partial \ln V}$. A plot of C_M vs. V can be found from wind tunnel tests. The derivative is then found as shown in Figure 11.

$$\frac{\partial C_M}{\partial \ln V} = \Delta C_M$$

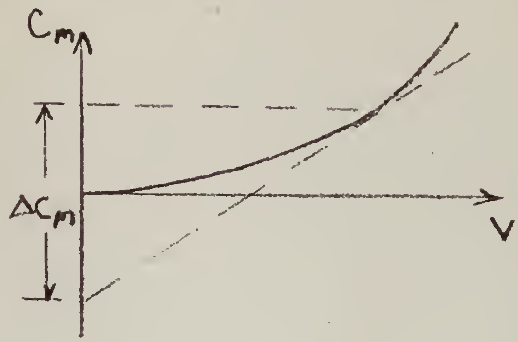


Figure 11.

E. Evaluation of the Longitudinal Modes Using Quasi-Static Concepts.

1. The Phugoid.

The damping factor, R_1, depends on the coefficient a_1 of equation B.2 and this quantity is,

$$a_1 = C_D \left(\frac{\partial \ln D}{\partial \ln V} - \frac{\partial \ln T}{\partial \ln V}\right) - C_{D_\alpha} \left(\frac{\frac{\partial C_M}{\partial \ln V}}{C_{M_\alpha}}\right) .$$

At relatively low velocity the approximate values of the derivatives are, $\frac{\partial \ln D}{\partial \ln V} = 2$, $\frac{\partial \ln T}{\partial \ln V} = -1$ (for a jet) and $\frac{\partial C_M}{\partial \ln V} = 0$. Thus the damping is,

$$R_1 = -\tfrac{1}{2}\left[C_D(3)\right] = -\frac{3}{2} C_D \quad \text{which is light.}$$

At higher speeds the effect of the last quantity in R_1 will be felt. This will decrease the damping since it will overcome the increase due to the other quantities.

The period factor, I_1, depends on both the coefficients of equation B.2. The coefficient, a_o, using the quasi-static stability at equilibrium, equation C.2, becomes,

$$a_o = \frac{C_L^2\left(2 + \frac{\partial \ln C_L}{\partial \ln V}\right)}{C_{M_\alpha}} \left.\frac{dC_M}{d\alpha}\right|_E$$

The period factor now becomes,

$$I_1 = \sqrt{C_L^2 \left(2 + \frac{\partial \ln C_L}{\partial \ln V}\right) \frac{C_{M_\alpha}\big|_E}{C_{M_\alpha}} - \frac{1}{4}\left(\frac{\partial \ln D}{\partial \ln V} - \frac{\partial \ln T}{\partial \ln V}\right)^2}$$

$$= C_L \sqrt{2 + \frac{\partial \ln C_L}{\partial \ln V}} \sqrt{\frac{C_{M_\alpha}\big|_E}{C_{M_\alpha}} - \frac{1}{4} \frac{\left(\frac{\partial \ln D}{\partial \ln V} - \frac{\partial \ln T}{\partial \ln V}\right)^2}{\left(2 + \frac{\partial \ln C_L}{\partial \ln V}\right)^2}} .$$

Again, at low speeds the approximations can be made that,

$$\left(\frac{C_D}{C_L}\right)^2 << 1 \qquad \text{and,} \qquad \frac{C_{M_\alpha}\big|}{C_{M_\alpha}} \doteq 1 .$$

So that,

$$I_1 = C_L \sqrt{2 + \frac{\partial \ln C_L}{\partial \ln V}} .$$

Now,

$$\omega = \frac{I_1}{t_s} = \frac{C_L\sqrt{2 + \frac{\partial \ln C_L}{\partial \ln V}}}{C_L\, V/g} = \frac{\sqrt{2 + \frac{\partial \ln C_L}{\partial \ln V}}}{V} \cdot g \quad .$$

The period is then,

$$T_p = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{2 + \frac{\partial \ln C_L}{\partial \ln V}}}\ \frac{V}{g}$$

and the wave length is,

$$L_p = V \cdot T_p = \frac{2\pi}{\sqrt{2 + \frac{\partial \ln C_L}{\partial \ln V}}}\ \frac{V^2}{q} \quad .$$

2. The Short Period Oscillation.

The damping factor is, $R_2 = -\frac{1}{2}\,(C_{L_\alpha} - C_{M_q}\, k_y^2)$.

Approximate values for these quantities are,

$$C_L = 5.0, \quad C_{M_q} = -0.7, \quad \text{and,} \quad C_{M_{\dot q}}\, k_y^2 = -5 \ .$$

The damping factor is the, $R_2 = -\frac{1}{2}\ 5 - (-5)\ = -5$ which indicates heavy damping.

The period or frequency factor was,

$$I_2 = \sqrt{-\frac{dC_M}{d\alpha}\bigg|_V k_y^2\mu - \frac{1}{4}(C_{M_\alpha} - C_{M_q}\, k_y^2)^2} \ .$$

Since, $-\frac{1}{4}\,(C_{L_\alpha} - C_{M_q}\, k_y^2)^2 = -\frac{1}{4}\,(C_{L_\alpha})^2 + \frac{1}{2}\, C_{L_\alpha} C_{M_q}\, k_y^2 - \frac{1}{4}\,(C_{M_q}\, k_y^2)^2$

and, $-\frac{dC_M}{d\alpha}\bigg|_V k_y^2\mu = -\, C_{M_\alpha}\, k_y^2\mu - C_{M_q}\, C_{L_\alpha}\, k_y^2$, the sum is,

$-\frac{1}{4}\,(C_{L_\alpha})^2 - \frac{1}{2}\, C_{L_\alpha}\, C_{M_q}\, k_y^2 - \frac{1}{4}\,(C_{M_q}\, k_y^2)^2 - C_{M_\alpha}\, k_y^2\mu$. Regrouping this quantity,

$$-\, C_{M_\alpha}\, k_y^2\mu - \frac{1}{4}\,(C_{L_\alpha} + C_{M_q}\, k_y^2)^2 \ .$$

And, using the values given before, it is seen that the item in the parenthesis is zero. At least the value in the parenthesis is very small and it being to the second power, divided by four, in addition to the presence of the large factor, μ, in the first term allows neglecting it in comparison with the firs

term. Thus there is now, $I_2 = \sqrt{C_{L_\alpha} (x - x_N) k_y^2 \mu}$, so that,

$$\omega = \sqrt{C_{L_\alpha} (x - x_N) k_y^2 \mu} / t_s \quad \text{and,} \quad t_s = \mu \frac{S^{1/2}}{V}$$

or,

$$\frac{1}{\omega} = \frac{\mu S^{1/2}}{\sqrt{C_{L_\alpha} (x - x_N) \frac{S}{i_y^2} \mu}} \frac{1}{V} .$$

The period is then, $T_R = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{\mu \, i_y^2 S}{C_{L_\alpha} (x - x_N) SV^2}}$ but, $\mu = \frac{2}{g\rho} \left(\frac{W}{S}\right)^{3/2} \frac{1}{W^{1/2}}$.

So then the period becomes,

$$T_R = 2\pi \sqrt{\frac{1}{C_{L_\alpha} (x - x_N)} \; \frac{2}{g\rho} \; \left(\frac{W}{S}\right)^{3/2} \; \frac{i_y^2}{S} \; \frac{W^{1/2}}{V^2}} ,$$

From the foregoing it is seen that depends on:

1(Center of gravity location: $\frac{1}{\sqrt{x - x_N}}$.

2) Altitude: $\frac{1}{\sqrt{\rho}}$.

3(Wing loading: $\sqrt[4]{(W/S(^3}$.

4) Size: $\sqrt[4]{W}$.

5) Mass Concentration: $\sqrt{\frac{i_y^2}{S}}$.

6) Velocity: $\frac{1}{V}$.

The decrement is seen to be,

$$- \frac{2\pi R_2}{I_2} = \pi \frac{(C_{L_\alpha} - C_{M_q} k_y^2)^2}{\sqrt{C_{L_\alpha} (x - x_N) k_y^2 \mu}} .$$

Although this analysis is specialized and the modes of motion are of such frequency and damping that they do not present much concern in design, the application of the quasi-static concepts has been shown. In regard to the importance of the modes, reference 8 points out that the importance of the phugoid should not be minimized too much.

The evaluation of the coefficients of the non-factored quartic, equation B.1, page 11, in terms of the quasi-static stability concepts will not be carried out. The coefficients have been arranged in a form for easy substitution of these quantities and a glance will show by what degree they can be simplified by such a substitution.

It is important, however, that a criterion for the separation of the quartic into modes of slow and fast frequency be established. If the ratio of the imaginary part of the roots describing the phugoid to that of the short period oscillation is small compared to unity, then the approximation is acceptable. From the simplified frequency factors used in the previous analysis, and, it is pointed out, the terms retained are the most important ones, it can be seen that this criterion is primarily dependent upon the factors $x - x_n$ and μ in the numerator of I_2. This follows since the other terms, although they may vary are relatively constant while the distance between the center of gravity and the neutral point may be anywhere from 0.005 to 0.05. The large variation of the density factor is shown in Appendix A.

The presentation thus far has dealt with the stick-fixed case only. The degree of freedom about the elevator hinge line is considered next.

F. Introduction of the Elevator Force.

In Section A it is recalled that the equation of longitudinal motion due to the degree of freedom about the elevator was locked, i.e., it implied that a change of elevator angle was impressed and held fast. Now this restriction is removed and a result obtained in the form of a modification to the original solution of the short period oscillation. This is the stick-free case. It is interesting to note here that the most common texts on this subject treat this case quite differently. Reference 4 rules out any modification of the phugoid mode and solves the remaining equations, dropping terms in u from an assumption of constant speed, for two short period modes. One of the latter is disregarded as the flapping of the elevator about its hinge line, while the other is discussed extensively as the mode causing the "porpoising" motion. Reference 1 writes off the freeing of elevators as a modification of the modes by a reduction of the tail effect by reason of the wing downwash and a possible contribution to the velocity moment derivative, M_u, by reason of some constant moment due to elevator mass or a spring attachment or friction.

The equation of motion resulting from this degree of freedom consists of equating the moments resulting from freeing the elevator to the inertial term, or moment of inertia of the elevator times the angular acceleration.

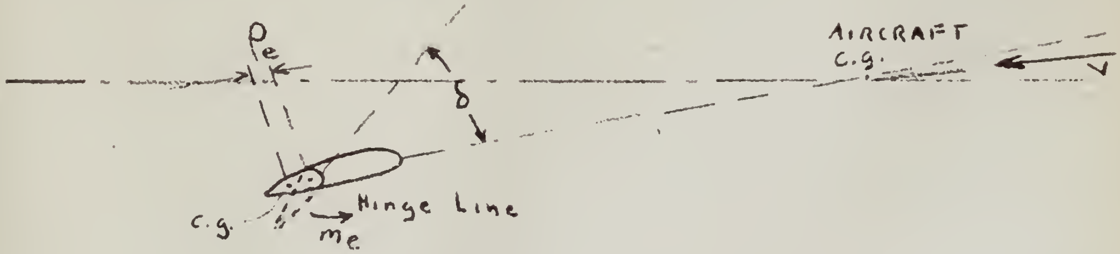


Figure 12.

Thus,

F.1 $$m_e \, i_e^2 \frac{d^2(\delta + \alpha + \gamma)}{dt^2} = M_\delta$$

and, $$M_\delta = -\frac{\partial C_L}{\partial \delta}\delta \rho \frac{V^2}{2} S_h C_h + m_e \rho_e V \frac{d\gamma}{dt} - m_e \rho_e r_h \frac{d^2(\alpha - \gamma)}{dt^2}.$$

From the figure it is seen that a down (positive) elevator deflection causes a nose down (negative) moment about the aircraft center of gravity. It was neglected for the stick-fixed case but will now be included. This quantity appears in equation A.6. There are effects in directions consistent with the other equations also but these are assumed small.

It is also reasonable to assume that there is an increase in moment about the hinge line of the elevator due to the change in angle of attack of the tail. As is pointed out in reference 4, this should be reduced to as near possible as possible. A typical fighter of the last World War had a value of C_{h_α} 25 o/o that of C_{h_δ}. C_{h_α} may be made to be small by proper design and many methods of reducing it are discussed in references 4 and 9. The incremental angle of attack of the tail itself is small for the portion of the flight history considered here, and even that is considerably reduced by wing downwash depending on the horizontal tail's position. The C_{h_α} term will be neglected here, but certainly should be included if design prevents its being made small. Any large value of C_{h_α} will detract from the pilot's "feel" of the aircraft. Derivatives of the hinge moment with respect to rates of change of α_t and δ are nelgected also.

As in reference 4, any modification of the phugoid from freeing the elevator will be assumed to be of no consequence. Thus the improvement to the stick-fixed approximation will only consider the short period oscillation, which, as mentioned before, assumes constant velocity. Thus this equation will neglect all changes with respect to velocity.

The second term in M_δ is the coriollis force which results from reference of the angular motion about the aircraft center of gravity to the center of gravity of the elevator. The last term is the mass moment effect due to the angular accleration about the aircraft center of gravity. So F.1 becomes, letting

$A = -\frac{\partial C_L}{\partial \delta} \rho \frac{V^2}{2} S_h C_h$ and non-dimensionalizing,

$$\frac{m_e i_e^2}{A t_s^2} \frac{d^2 \delta}{d(\frac{t}{t_s})^2} + \delta - \frac{m_e \rho_e V}{A t_s} \cdot \frac{d \gamma}{d(\frac{t}{t_s})^2} + \frac{m_e(\rho_e r_h + i_e^2)}{A t_s^2} \frac{d^2(\alpha + \gamma)}{d(\frac{t}{t_s})^2} = 0 .$$

Now let,

$$t_e = \frac{m_e i_e^2}{A t_s^2} = \frac{m_e i_e^2 2V^2}{-C_{h\delta} S_h C_h \rho V^2 \mu^2 S} = \frac{m_e i_e^2 S^{1/2}}{-C_{h\delta} \mu m S_h C_h}$$

$$\Delta = \frac{\rho_e r_h}{i_e^2} \qquad \text{and,} \qquad B = \frac{\rho_e S^{1/2}}{i_e^2}$$

then F.1 becomes,

$$\text{F.1} \qquad t_e(1 + \Delta)\ddot{\gamma} - t_e B \mu \dot{\gamma} + t_e(1 + \Delta)\ddot{\alpha} + t_e \ddot{\delta} + \delta = 0.$$

This equation along with equations A.5 and A.6 of Section A gives the system which shows the modes of motion for the stick-free case. The complete system of equations, including A.4, should be considered for a rigorous analysis of the modes of motion. However, as was discussed previously, the short period oscillation presumes constant velocity and this is again assumed. Thus there is the following set of equations.

A.5	$\dot{\gamma}$	$-C_{L_\alpha} \alpha$	0	$= 0$
A.6	$\ddot{\gamma} - m_q \dot{\gamma}$	$\ddot{\alpha} - m_q \dot{\alpha} - m_\alpha \mu \alpha$	$- m_\delta \mu \delta$	$= 0$
F.1	$t_e(1+\Delta)\ddot{\gamma} - t_e B \mu \dot{\gamma}$	$t_e(1 + \Delta)\ddot{\alpha}$	$t_e \ddot{\delta} + \delta$	$= 0$

where, $m_q = C_{M_q} k_y^2$; $m_\alpha = C_{M_\alpha} k_y^2$; $m_\delta = C_{M_\delta} k_y^2$.

Assuming a solution of the form, $x = x_o e^{zt}$, there is the determinant,

$$\begin{vmatrix} z & -C_{L_\alpha} & 0 \\ z^2 - m_q z & z^2 - m_q z - m_\alpha \mu & - m_\delta \mu \\ t_e(1 + \Delta)z^2 - t_e B\mu z & t_e(1 + \Delta)z^2 & t_e z^2 + 1 \end{vmatrix} = 0 ,$$

or,

$$(t_e z^2 + 1)(z^3 - m_q z^2 - m_\alpha\big|_V \mu z)$$

$$+ t_e \left\{(1 + \Delta)z^2 \left[C_{L_\alpha} m_\delta \mu \quad m_\delta \mu z\right] - B\mu z\, C_{L_\alpha} m_\delta \mu\right\} = 0$$

where recall,

$$m_\alpha \Big|_V = k_y^2 \left(\frac{\partial C_M}{\partial \alpha} + \frac{\partial C_M}{\partial q} \frac{\frac{\partial C_L}{\partial \alpha}}{\mu}\right) ,$$

which is, regrouping,

$$z^3 + z^2(C_{L_\alpha} - m_q) + z(-m_\alpha\big|_V)\mu + t_e \left\{z^2 \left[z^3 + z^2(C_{L_\alpha} - m_q) - z m_\alpha\big|_V \mu\right]\right.$$

$$\left. + (1 + \Delta)z^2(C_{L_\alpha} + z)m_\delta \mu - B\mu\, m_\delta \mu C_{L_\alpha} z \right\} = 0$$

or, eliminating the root, $z = 0$, and letting, $f_1(z) = z^2 + z(C_{L_\alpha} - m_q)$
$- m_\alpha\big|_V \mu = 0$

and, $f_2(z) = (1 + \Delta)(z^2 + C_{L_\alpha} z) - C_{L_\alpha} B\mu$ there is,

$$F(z,t_e) = f_1(z) \quad t_e\left[z^2 f_1(z) + m_\delta \mu f_2(z)\right] = 0.$$

If now the case is considered for the mass density, μ , large, which is usually the case except for gliders, then the quantity, t_e, essentially vanishes. So, approximately then, $f_1(z) = 0$. The entire expression, $F(z,t_e)$, may be analyzed from the point of view of using the roots, z_1, of $f_1(z)$ in the evaluation.

The effect of freeing the elevators then will be the determination of the damping factor and frequency of the short period oscillation from a factor, Δz, to be determined.

Reviewing the situation, the characteristic equation obtained from the determinant gives a fifth degree equation in z. Elimination of the null root gives $F(z,t_e)$ which is fourth degree in z. Reference 4 then solves this equation for two pairs of roots which give the two short period oscillations as previously mentioned.

There is then, in mathematical terms,

$$t_e = 0 \qquad \text{giving,} \qquad f_1(z) = 0 \qquad \text{where,} \qquad z = z_1$$

then

$$\frac{dz}{dt_e} = -\frac{\frac{\partial F}{\delta t_e}}{\frac{\partial F}{\delta z}} = -\frac{m_\delta \mu\, f_2(z_1)}{f_1'(z_1)}$$

$$= -m_\delta \mu \frac{(1 + \Delta)z_1^2 + (1 + \Delta)\frac{\partial C_L}{\delta \alpha} z_1 - \frac{\partial C_L}{\delta \alpha} B \mu}{2z_1 + (\frac{\partial C_L}{\delta \alpha} - m_q)} .$$

Now,

$$z_1 = R_1 + iI_1 \qquad\qquad z_1^2 = R_1^2 - I_1^2 + 2i\, R_1\, I_1$$

but,

$$z_1 = -\frac{(C_{L_\alpha} - m_q)}{2} \pm \frac{i\sqrt{-4m_\alpha|_V \mu - (C_{L_\alpha} - m_q)^2}}{2}$$

so,

$$-2R_1 = C_{L_\alpha} - m_q \quad ; \quad I_1 = \sqrt{-m_\alpha|_V \mu - \frac{(C_{L_\alpha} - m_q)^2}{4}}$$

and,

$$R_1^2 + I_1^2 = \frac{(C_{L_\alpha} - m_q)^2}{4} - m_\alpha|_V \mu - \frac{(C_{L_\alpha} - m_q)^2}{4} = -m_\alpha|_V \mu$$

therefore,

$$\frac{dz}{dt_e} = -m_\delta \mu \frac{(1 + \Delta)(R_1^2 - I_1^2 + 2i\, R_1\, I_1 + (R_1 + iI_1)C_{L_\alpha}) - C_{L_\alpha} B \mu}{2R_1 + 2iI_1 - 2R_1}$$

$$= -m_\delta \mu \Bigg[\frac{(1 + \Delta)(2i\, R_1\, I_1 + iI_1\, C_{L_\alpha})}{2i\, I_1} - i^2 \frac{(1 + \Delta)\left[2R_1^2 - (R_1^2 + I_1^2) + R_1\, C_{L_\alpha}\right] - C_{L_\alpha} B \mu}{2i\, I_1} \Bigg]$$

$$= -m_\delta \mu \left[\frac{(1 + \Delta)(2R_1 + C_{L_\alpha})}{2} \right]$$

$$+ i\, m_\delta \mu \left\{ \frac{(1 + \Delta)\left[2R_1^2 - (R_1^2 + I_1^2) + R_1\, C_{L_\alpha}\right] - C_{L_\alpha} B \mu}{2I_1} \right\}$$

but,

$$2R_1^2 + C_{L_\alpha} = -C_{L_\alpha} + m_q \quad C_{L_\alpha} = m_q$$

and

$$2R_1^2 + R_1 C_L = \frac{C_{L_\alpha}^2}{2} - C_{L_\alpha} m_q + \frac{m_q^2}{2} - \frac{C_{L_\alpha}^2}{2} + \frac{C_{L_\alpha} m_q}{2}$$

$$= -\frac{m_q}{2}\left(C_{L_\alpha} - m_q\right)$$

therefore,

$$\frac{dz}{dt_e} = -m_\delta \mu \left\{ \frac{(1+\Delta)m_q}{2} + i\,\frac{(1+\Delta)\left[m_\alpha |_V \mu + \frac{m_q}{2}\left(\frac{\partial C_L}{\partial \alpha} - m_q\right)\right] + C_{L_\alpha} B \mu}{2I_1} \right\}$$

but now, $\Delta z = \frac{dz}{dt_e} t_e$. Considering the term, $-m_\delta \mu t_e$,

$$-m_\delta \mu t_e = \frac{-C_{m_\delta} \mu m_e i_e^2 S^{1/2} k_y^2}{-C_{h_\delta} \mu m S_h C_h}$$

but,

$$C_{m_\delta} \rho \frac{V^2}{2} S^{3/2} = -\frac{\partial C_{L_h}}{\partial \delta} \rho \frac{V^2}{2} S_h r_h$$

so,

$$C_{M_\delta} = -\frac{\partial C_{L_h}}{\partial \delta} \frac{S_h r_h}{S^{3/2}}$$

thus,

$$-m_\delta \mu t_e = \frac{\frac{\partial C_{L_h}}{\partial \delta} m_e i_e^2 r_h}{-\frac{\partial C_h}{\partial \delta} m S C_h}$$

consequently,

$$\Delta z = -\frac{\frac{\partial C_{L_h}}{\partial \delta} m_e i_e^2 r_h}{\frac{\partial C_h}{\partial \delta} m S C_h} \left\{ \left(1 + \frac{\rho_e r_h}{i_e^2}\right) C_{m_q} \frac{k_y^2}{2} + i\,\frac{\left(1 + \frac{\rho_e r_h}{i_e^2}\right)\left[m_\alpha \mu + \frac{m_q}{2}\left(C_{L_\alpha} + m_q\right)\right] + C_{L_\alpha} B \mu}{2I_1} \right\}.$$

It is seen that, since both C_{h_δ} and C_{m_q} are negative, the real part, which indicates damping, is negative and thus this motion is damped for this acceptance of sign. However, the value of C_{h_δ} might go positive. For instance, reference 9 shows that a horizontal tail design using an NACA 66-009 airfoil secti

with flap 0.30 chords in length, an air gap of 0.005 chords and with the hinge line at a positon of 0.50 flap chords, reduces the C_{h_α} to zero between $\alpha = -8^\circ$ to $\alpha = 5^\circ$, while C_{h_δ} is positive. This reference also shows that C_{h_δ} may go positive depending on the use or non-use of a horn balance. The use of a horn balance, in general, appears to make C_{h_α} go slightly positive also, the magnitude being very small

Thus this mode of motion may be positively damped or possess moderate divergence. The frequency is given by,

$$\frac{\left(1 - \frac{\rho_e r_h}{i_e^2}\right)\left[m_\alpha \mu + \cdots\right] + C_L \frac{\rho_e s^{1/2}}{i_e^2} \mu}{2\sqrt{-m_\alpha \mu \quad \cdots}}$$

where the term neglected is, $(C_{L_\alpha} + m_q)$, and is considered small as in the previous discussion. The terms in the brackets are negative, the last term positive. The total value of the quantity is not too large.

It is difficult to strike a concrete comparison of the mode of motion solved for here and those found in reference 4. This solution does not solve the equation by a factorization but by the assumption of the quantity, t_e, being large. In general, the roots solved for by Dr. Scheubel are not the same as those causing the porpoising motion described in reference 4, since there it is indicated that the nature of the porpoising mode is a function of C_{h_α} and C_{h_δ}. Here, it is dependent upon C_{h_δ}, $\frac{\partial C_{L_h}}{\partial \delta}$ and C_{m_q}. It is not to be implies that this is not such a type of motion, rather that it is not the same implicitly.

Dr. Scheubel neglected the quantity C_{h_α}, it is believed, due to the fact that in Germany it was the practice to design the controls such that it was reduced to a minimum. He discussed the handling qualities of a very large four-engined flying boat in which the stick forces per g were very small without power boost, the exact values of which are not remembered. In regard to

lateral control, however, it is remembered that he stated that one person could maintain straight and level flight with both engines dead on one side with one foot on the rudder control, again without power boosted controls.

Inclusion of the C_{h_α} term would place it in the solution of $f_1(z)$ so that it would detract or add to both the damping and frequency terms as it was negative or positive in the ratio of $\frac{C_{h_\alpha}}{C_{h_\delta}}$ Furthermore, in as much as downwash would multiply the value of the variable, α , which affects it by a factor less than unity, this would further decrease the ratio itself.

G. Effect of Elevator Impulse Type Forcing Function.

Reiterating, it is recalled that the stick-fixed case involved the importance of an elevator displacement or a gust and the ensuing motion was analyzed in the light of the elevator being locked in position. The stick-free case allowed the elevator to float freely about its hinge line. Furthermore, the equations of motion were assumed homogeneous. An additional case is now considered in which the equations involve a forcing function and further this forcing function is an elevator impulse, an elevator deflection of short duration. This case is analogous to the incorporation in the system of an on-off type autopilot.

An elevator impulse as shown in Figure 13 is imposed. This impulse is given the symbol, I_δ , and is,

$$I_\delta = \int_0^\tau \delta \, d\tau$$

or,

$$I_\delta \cdot t_s = \int_0^t \delta \, dt$$

The first phase following the impulse, that of the short-period oscillation, is considered and thus there are the equations,

Figure 13.

$$\text{A.5} \qquad \dot{\gamma} \qquad - \frac{\partial C_L}{\partial \alpha}\alpha \qquad = \frac{\partial C_L}{\partial \delta}\delta(\tau)$$

$$\text{A.6} \qquad \ddot{\gamma} - m_q\dot{\gamma} \qquad + \ddot{\alpha} - m_q\dot{\alpha} - m_q\mu\alpha \qquad = m_\delta \mu \delta(\tau).$$

The effect of the impulse on the equation of motion perpendicular to the wind axis is negligible for conventional aircraft since the only force created is a lift force, which, due to the short duration of the impulse, does not act long enough to be effective. This is not so for tailless aircraft, however, since for such aircraft the deflection of the elevator, which is actually a portion of the main lifing foil trailing edge, changes the wing angle of attack and causes an appreciable effect in this direction.

The effect of the impulse on the equation of moments causes an initial angular velocity which will be designated $\hat{\omega}_o$, but no angular displacement. If there is no disturbance in flight path angle, which is nearly correct for conventional aircraft, this initial angular velocity will consist essentially of solely a time rate of change of angle of attack. This is so because of the mass distribution of a conventional aircraft compared to, say, a tailless aircraft. Thus initially, at $\hat{\tau} = 0$; $\gamma_o = \dot{\gamma}_o = 0$; $\alpha_o = 0$; $\dot{\alpha} = \hat{\omega}_o$

Solving A.5 for $\dot{\gamma}$, substituting in A.6, there is then for the short period oscillation,

$$\ddot{\alpha} - 2R_1\dot{\alpha} + (R_1^2 + I_1^2)\alpha = \ddot{\alpha} + \dot{\alpha}(C_{L_\alpha} - C_{m_q} k_y^2) + \alpha\left[-\mu k_y^2(C_{m_\alpha}|_V)\right] = 0.$$

The solution for α is,

$$\alpha = e^{R_1\hat{\tau}}(A_1 \cos I_1\hat{\tau} + B_1 \sin I_1\hat{\tau}).$$

Imposing the first initial condition gives, $A_1 = 0$. Differentiating α then,

$$\dot{\alpha} = \hat{\omega}_o = e^{R_1\hat{\tau}}(R_1 B_1 \sin I_1\hat{\tau} + I_1 B_1 \cos I_1\hat{\tau}).$$

The second initial condition gives, $B_1 = \frac{\hat{\omega}_o}{I_1}$. The Principle of Impulse and Momentum states that the change in angular momentum about any axis equals the angular impulses of the applied forces about the axis, or,

$m\, i_y^2\, \hat{\omega}_o = m_\delta\, \delta(t)$ so that,

$$\hat{\omega}_o = \frac{\partial C_L}{\partial \delta}\, \frac{\rho \frac{V^2}{2} S^{3/2}\, t_s}{m\, i_y^2} \int_0^{\hat{\tau}} \delta\, d\hat{\tau}$$

$$= \frac{C_{m_\delta}\, V^2\, S\, t_s^2}{\frac{2m}{\rho S^{3/2}}\, i_y^2\, S}\, I_\delta$$

$$= C_{m_\delta}\, \frac{S}{i_y^2}\, \frac{1}{\mu}\, \frac{V^2}{S}\, \frac{\mu^2 S}{V^2}\, I_\delta = m_\delta\, \mu\, I_\delta$$

This evaluates the constant, B_1, so that consequently,

$$\alpha = \frac{m_\delta \mu}{I_1} I_\delta \, e^{R_1 \tau} \sin I_1 \tau \quad .$$

From equation A.5 there is, $\dot{\gamma} = C_{L_\alpha} \alpha = C_{L_\alpha} \frac{m_\delta \mu}{I_1} I_\delta \, e^{R_1 \tau} \sin I_1 \tau$. Integrating,

$$\text{(G.1)} \quad \gamma = \frac{m_\delta \mu \, C_{L_\alpha}}{I_1} I_\delta \, \frac{e^{R_1 \tau}}{R_1^2 + I_1^2} \left(R_1 \sin I_1 \tau - I_1 \cos I_1 \tau \right) + \gamma_\infty$$

Since at, $\tau = 0$, $\gamma = 0$ there is finally,

$$\text{G.1} \quad \gamma = \frac{m_\delta \mu \, C_{L_\alpha}}{R_1^2 + I_1^2} \, I_\delta \left[1 - e^{R_1 \tau} \left(\cos I_1 \tau - \frac{R_1}{I_1} \sin I_1 \tau \right) \right] .$$

This initial phase of the response to an impulse type forcing function is depicted qualitatively in Figure 14. In equation G.1 the values of R_1 and I_1 are the same as those found in Section F., page 36. Thus the transient is heavily damped and typical values show it to have reached the steady state in one-half periods.

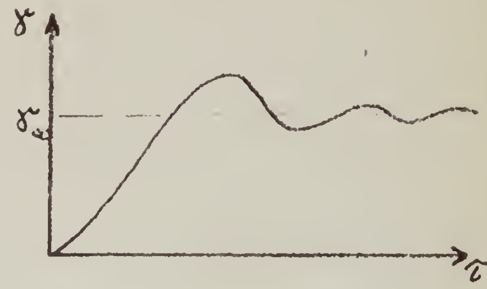


Figure 14.

Examining further the steady state of G.1,

$$\gamma_\infty = \frac{m_\delta \mu \, C_{L_\alpha} \, I_\delta}{R_1^2 + I_1^2} = \frac{C_{m_\delta} k_y^2 \mu \, C_{L_\alpha} \, I_\delta}{-C_{m_\alpha}|_V \, k^2 \mu}$$

and letting, $C_{m_\alpha}|_V = -C_{L_\alpha}(x - x_N|_V)$ there is, $\gamma_\infty = \dfrac{C_{m_\delta} \, I_\delta}{x - x_N|_V}$ If the denominator is small, the response is very sensitive to the impulse. Dr Scheubel cited an example of neutral stability where the steady state then becomes,

$$\gamma_\infty = \frac{C_{m_\delta} \, I_\delta}{- C_{m_q} \dfrac{1}{\mu}} \qquad \text{since,} \qquad x = x_N \quad .$$

Now, $C_{m_\delta} = - \dfrac{\partial C_{\ell h}}{\partial \delta} \dfrac{S_h \, r_h}{S^{3/2}}$ and, $- C_{m_q} \dfrac{1}{\mu} = \dfrac{\partial C_{\ell h}}{\partial \alpha_t} \dfrac{S_h \, r_h^2}{S^2} \dfrac{1}{\mu}$. Using values of,

$V = 450$ mph; $S = 270$ ft.^{2}; $I_\delta = -1^\circ$; $t_s = 0.2$ sec.; $\mu = 100$, the response is,

$$\gamma_\infty = - \frac{\frac{\partial C_{\ell h}}{\partial \delta}}{\frac{\partial C_{\ell h}}{\partial \alpha_t}} \cdot \frac{S^{1/2}}{r_h} \cdot \frac{V}{S^{1/2}} \cdot I_\delta \cdot t_s = -0.7 \times 1 \times \frac{660}{16.5} \times - 0.0035 = 5.7^\circ$$

which is appreciable for such a small impulse. It is pointed out that this value of elevator effectiveness, $\frac{d\alpha_h}{d\delta}$, corresponds to an elevator area to total horizontal tail area ratio of 0.6 according to an empirically determined curve in reference 4.

The second phase of the motion following the impulse is that of the flight path oscillation. Thus equations A.4 and A.5 are used. These should be modified as before for the variable, α. However, it will be assumed that the derivative, $\frac{\partial C_M}{\partial \ln V}$, is approximately zero and will be neglected.

A.4 $$\dot{u} - C_D\left(\frac{\partial \ln D}{\partial \ln V} - \frac{\partial \ln T}{\partial \ln V}\right) u + C_L \gamma = f_u(\tau)$$

A.5 $$- C_L\left(2 + \frac{\partial \ln C_L}{\partial \ln V}\right)u \qquad + \dot{\gamma} \quad = f_\gamma(\tau) .$$

From the past analysis of the two modes, the relative magnitudes of their periods, and the magnitudes of their damping factors, it can be assumed that the response to the impulse of the rotary motion has reached the steady state condition by the time the flight path response takes effect. So the initial conditions are, at, $\tau = 0$: $\gamma = \gamma_\infty$, and $u = 0$, and $\dot{\gamma} = 0$.

Again the forcing function creates no accountable effect either along the wind axis or perpendicular to it. Thus a good approximation is to work with homogeneous equations. So,

$$\ddot{\gamma} - 2R_2 \dot{\gamma} + (R_2^2 + I_2^2)\gamma \doteq \ddot{\gamma} - C_D\left(\frac{\partial \ln D}{\partial \ln V} - \frac{\partial \ln T}{\partial \ln V}\right)\dot{\gamma} + C_L^2\left(2 + \frac{\partial \ln C_L}{\partial \ln V}\right)\gamma = 0$$

A solution for γ is, $\gamma = e^{R_2 \tau}(A_2 \cos I_2 \tau + B_2 \sin I_2 \tau)$. The first initial condition gives, at $\tau = 0$, $A_2 = \gamma_\infty$. The second, $B_2 = - \frac{R_2}{I_2}\gamma_\infty$ since, $\dot{\gamma} = 0 = R_2 A_2 + I_2 B_2$. So now,

$$\text{G.2} \qquad \gamma = \gamma_\infty e^{R_2 \tilde{\tau}} \left(\cos I_2 \tilde{\tau} - \frac{R_2}{I_2} \sin I_2 \tilde{\tau} \right)$$

The inclination of the flight path, γ, may be defined, using Figure 15, as,

$$\gamma = \frac{1}{V} \frac{dh}{dt}$$

Figure 15.

and, putting this in terms of the common time, $\tilde{\tau}$, since, $t_s = \mu \frac{S^{1/2}}{V} = \frac{\ell_s}{V}$ where. $\ell_s = \mu S^{1/2}$ so that,

$$\gamma = \frac{d \frac{h}{\ell_s}}{d \tilde{\tau}}$$ and γ is of the form, $\gamma = e^{R_2 \tilde{\tau}} (A_2 \cos I_2 \tilde{\tau} + B_2 \sin I_2 \tilde{\tau})$. The first initial condition gives, at $\tilde{\tau} = 0$, $\gamma_\infty = A_2$. The second, $\dot{\gamma} = 0 = R_2 A_2 + I_2 B_2$ so, $B_2 = -\frac{R_2}{I_2} \gamma_\infty$. So now,

$$\text{G.2} \qquad \gamma = \gamma_\infty e^{R_2 \tau} \left(\cos I_2 \tilde{\tau} - \frac{R_2}{I_2} \sin I_2 \tilde{\tau} \right)$$

The inclination of the flight path, , may be defined, using Figure 15, as,

$$\gamma = \frac{1}{V} \frac{dh}{dt}$$

and, making this quantity non-dimensional by use of, $t_s = \mu \frac{S^{1/2}}{V} = \frac{\ell_s}{V}$ where $\ell_s = \mu S^{1/2}$, so now,

$$\text{G.3} \qquad \frac{h}{\ell_s} = \int_0^{\tilde{\tau}} \gamma \, d\tilde{\tau} = \frac{\gamma_\infty}{I_2} \left\{ e^{R_2 \tilde{\tau}} \sin I_2 \tilde{\tau} \frac{I_2^2 - R_2^2}{R_2^2 + I_2^2} + \frac{2 I_2 R_2}{R_2^2 + I_2^2} \left(1 - e^{R_2 \tilde{\tau}} \cos I_2 \tilde{\tau} \right) \right\}$$

The maximum altitude gained following an impulse is then,

$$h = \frac{\ell_s}{I_2} \gamma_\infty = \frac{\ell_s}{C \sqrt{2 + \frac{\partial \ln C_L}{\partial \ln V}}} \gamma_\infty \doteq \frac{V^2}{g\sqrt{2}} \gamma_\infty .$$

This result may be obtained by a slightly different approach. Considering equation (B) above,

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$$u = \frac{\dot{\gamma}}{C_L\left(2 + \frac{\partial \ln C_L}{\partial \ln V}\right)} \doteq \frac{C_L \cdot \dot{\gamma}}{R_2^2 + I_2^2}$$

since from Section IV,B,

$$R_2^2 + I_2^2 \doteq C_L^2\left(2 + \frac{\partial \ln C_L}{\partial \ln V}\right).$$

From before,

$$\dot{\gamma} = -\gamma_\infty e^{R_2 \tau} \frac{R_2^2 + I_2^2}{I_2} \sin I_2 \tau$$

therefore,

$$u = -\gamma_\infty \frac{C_L}{I_2} e^{R_2 \tau} \sin I_2 \tau$$

and considering maximum u,

$$u_{max} = \frac{-\gamma_\infty C_L}{C_L\sqrt{2 + \frac{\partial \ln C_L}{\partial \ln V}}} \doteq -\frac{\gamma_\infty}{\sqrt{2}}$$

and, neglecting energy loss due to damping,

$$Hk_0 + h + \frac{u^2}{2g} = \text{constant}.$$

Substituting for γ and integrating,

$$\text{G.3} \quad \frac{h}{\ell_s} = \int_0^{\tau} \gamma \, d\tau = \frac{\gamma_\infty}{I_2}\left\{ e^{R_2 \tau} \sin I_2 \tau \frac{I_2^2 - R_2^2}{R_2^2 + I_2^2} - \frac{2R_2 I_2}{R_2^2 + I_2^2}\left(1 - e^{R_2 \tau} \cos I_2 \tau\right)\right\}.$$

The maximum altitude gained following an impulse is then, assuming R_2 and $\frac{\partial \ln C_L}{\partial \ln V}$ small,

$$h \doteq \frac{\ell_s}{I_2}\gamma_\infty \doteq \frac{\ell_s}{C_L\sqrt{2 + \frac{\partial \ln C_L}{\partial \ln V}}}\gamma_\infty \doteq \frac{V^2}{g\sqrt{2}}\gamma_\infty .$$

This result may be obtained by a slightly different approach. Considering equation A.5 above,

$$u = \frac{\dot{\gamma}}{C_L\left(2 + \frac{\partial \ln C_L}{\partial \ln V}\right)} \doteq \frac{C_L \cdot \dot{\gamma}}{R_2^2 + I_2^2}$$

Differentiating G.2,

$$\dot{\gamma} = -\gamma_\infty e^{R_2\tau} \frac{R_2^2 + I_2^2}{I_2} \sin I_2 \tau .$$

Therefore,

$$u = -\gamma_\infty \frac{C_L}{I_2} e^{R_2\tau} \sin I_2 \tau .$$

Considering maximum u,

$$u_{max} = \frac{-C_L \gamma_\infty}{C_L \sqrt{2 - \frac{\partial \ln C_L}{\partial \ln V}}} \doteq -\frac{\gamma_\infty}{\sqrt{2}}$$

and, neglecting energy loss due to damping,

$$H_o + h + \frac{V^2}{2g} = \text{constant}$$

so that, $d(h + \frac{V^2}{2g}) = 0$ or, $dh = -\frac{V^2}{g} \frac{dV}{V} = -\frac{V^2}{g} u$ and, finally,

$|\Delta h| = \frac{V^2}{g} |u| = \frac{V^2}{g} \frac{\gamma_\infty}{\sqrt{2}}$ which is as before.

Using the same figures as before, the maximum gain in altitude very approximately becomes,

$$\Delta h = \frac{(660)^2}{g\sqrt{2}} \times 0.0995 = 950 \text{ feet}$$

Discussion and Conclusions

The quasi-static stability concepts based on equilibrium of forces and moments at various phases of the flight history of an aircraft following a disturbance appear to be sound for an acceptance of the description of the flight history stated earlier. Neglecting additional effects such as elasticity, these concepts account for the apparent increase in stability due to compressibility effects associated with the aircraft drag, lift and moment. If the values of these quasi-static stability criteria could be ascertained individually and in their entirety, presumably they would include interference effects. As it is, no American literature known to the writer discusses longitudinal stability comparisons at low and high speeds. Dr. Scheubel discusses the comparison with the results shown in Figures 5 and 6. This was apparently based on results of flight tests or the inclusion of the additional factors in the quasi-static criteria analytically determined. This latter would necessarily neglect interference effects.

For the special case of separation of the motion into the phugoid and short period modes good agreement with standard developments was had neglecting the additional factors mentioned above.

The modification of the phugoid mode due to freeing the elevators was neglected as is usually done. The resulting equation was analyzed for the special case of the aircraft mass density factor of large magnitude. This assumption was used to approximate only one pair of the complex conjugate roots. Comparison with standard references showed this motion to be approximately the "porpoising" mode although not implicitly.

Of some interest is the German use of the quantity $S^{\frac{1}{2}}$, square root of the wing area, as the non-dimensionalizing length term in conversion of forces and

moments to coefficients. Actually, Dr. Scheubel did not point this out as having wide German usage, but rather as being used at Darmstadt Technical Institute. It might be pointed out that the concept of aspect ratio would be lost by use of this length. This might be better than the nebulous definition of mean aerodynamic chord present in current American aerodynamic literature For instance, reference 4, supposedly a fairly accredited text, defines mean aerodynamic chord as 1) an average chord, for a finite wing of varying section, which, when multiplied by the average section moment coefficient, dynamic pressure and wing area, gives the moment for the entire wing; 2) the chord of an imaginary airfoil which throughout the normal flight range has the same force vectors as the three-dimensional wing and can be determined from wind tunnel tests or a graphical analysis which assumes a special shape to the wing plan, and 3) a chord on which all the forces and moments acting on the surface can be represented as acting and on which the pitching moment coefficient is invariant with lift coefficient. Dr. Scheubel pointed out that the special reason for the use of this quantity was the fact that the ratio of the square root of the wing area to the tail length was found to range between 0.8 and 1.25 and the average value found for some ninety aircraft was 1.01 The advantage of approximating this ratio to unity resulted in the adoption of this length.

The extension of such a method to the lateral or asymmetric case is discussed in general terms in Appendix B.

The development for the longitudinal case as a whole does not consider all the factors bearing on the problem by any means. It was pointed out by Dr Scheubel that this was a good approximation from the project engineer's viewpoint and must necessarily be improved as the design develops. For instance. the effects of downwash and the influence of the powerplant have a large effect on the damping term. In any case it is a slightly different view of the problem and its worth rests on that.

APPENDIX A

The mass number, μ , as mentioned in reference 1 was first suggested by Glauert for use in the non-dimensionalizing of the equations of motion That it also follows from dimensional analysis, e.g., the Pi Theorem, is well known Due to its extensive use in the literature, a discussion of it here is only warranted by the slightly different definition of it in this paper and the fact that considerations of its magnitude for use in the approximations in the paper, necessitate recalling what factors bear on its magnitude.

The difference in definition lies in the use of the mean wing chord as the representative length in references 1 and 4. Furthermore, the mass or density number used in this paper is twice that used in these references

The quantity then is written as,

$$\mu = \frac{2W}{\rho g S^{3/2}} = \frac{2}{g\rho} \left(\frac{W}{S}\right)^{3/2} \left(\frac{1}{W}\right)^{1/2}$$

from which it is seen to depend on altitude, ρ , on wing loading, W/S, and on weight, 1/W. The tendencies with these parameters varying are as follows;

a. Altitude - As altitude increases, μ increases. An altitude increase from sea level to 40,000 feet causes an increase in μ of from 1 to 4

b. Wing Loading - Since it is to the 3/2 power, an increase of wing loading of 1:2 gives an increase in μ of 1:2.83.

c. Weight - As weight increases, μ decreases as $(1/W)^{\frac{1}{2}}$. In this case a 1:2 increase in weight give a 1:0.7 decrease in μ . Typical values of μ are:

Large gliders at low altitude: $\mu \doteq 5\text{-}10$

Small fighters at high altitude: $\mu \doteq 500$

Large commercial planes: $\mu \doteq 50\text{-}100$

APPENDIX B

Discussion of the Extension of Quasi-Static Stability to the Lateral or Asymmetric Motion

The possibility of the extension of the quasi-static stability concepts to the asymmetric motion of an aircraft seems to be remote insofar as the same line of reasoning be followed. It will be recalled that in both quasi-staiic stability determinations consideration was made of forces in a single linear direction and of moments about a point in one plane. In effect the ratio of changes in parameters affecting both systems lead to the quasi-static stability concepts. Of fundamental importance was the fact that consideration of those changes did not affect the additional linear force equation. Such a correspondence does not hold in the lateral case..

Assuming a conventional aircraft it has been found that the modes of motion in the lateral case are four in number. Following the same line of reasoning as was done in the longitudinal case, a disturbance could, all factors considered, cause a linear displacement laterally, i.e., along the y-axis, a rolling displacement, and a yawing displacement. References 4, 10 and 11 show that the solution of the lateral equations results in a characteristic quartic which gives roots λ_1, λ_2, λ_3, and λ_4. In solving for these roots all neglect the presence of products of inertia, that is, they assume that the principal longitudinal axis of the aircraft in is line with the flight path, and assume level flight or $\tan\theta = 0$. The moment composed of three components given by:

I $\quad \beta_1 = B_1 e^{\lambda_1 \tau}$; $p_1 = P_1 e^{\lambda_1 \tau}$; $r_1 = R_1 e^{\lambda_1 \tau}$

II $\beta_2 = B_2 e^{\lambda_2 \tau}$; $p_2 = P_2 e^{\lambda_2 \tau}$; $r_2 = R_2 e^{\lambda_2 \tau}$

III $\beta_3 = B_3 e^{\lambda_3 \tau}$ $B_4 e^{\lambda_4 \tau}$; $p_3 = P_3 e^{\lambda_3 \tau} + P_4 e^{\lambda_4 \tau}$; $r_3 = R_3 e^{\lambda_3 \tau}$ $R_4 e^{\lambda_4 \tau}$

where the quantities β, p, and r are indicative of sideslip, angular velocity of roll, and angular velocity of yaw respectively. The coefficients P_i, R_i and B_i are dependent on initial conditions, only four being arbitrary, the remainder are determined from these. These individual motions will be considered separately.

Considering the first group, the root, λ_1, is small and usually positive giving light divergence. This is the so-called spiral motion and either a small disturbance which is pure sideslip, roll, or yaw will develop an interdependence of both rolling and yawing moments on both sideslip angle and yawing velocity. Following quasi-static reasoning, since the period is long, this motion indicates the establishment of a unique relationship between initial disturbance increments and resulting forces and/or moments in equilibrium after a long period of time. This does not follow since either aileron, rudder or sideslip, due to a gust, say, may cause the motion and all facets of lateral motion are involved in the resulting motion.

The second group is characterized by a large root, λ_2, usually negative. B_2 and R_2 are small in comparison with P_2 so that the motion is heavily damped and is essentially pure roll. No known reference considers this motion of any importance. Reference 12 which includes the effect of finite product of inertia and initial inclination of the flight path excludes any mention of it also. Reference 13, which is an investigation of unsatisfactory lateral stability on an actual current combat aircraft, does not mention it. This evidence coupled with the fact that the motion involves only one degree of freedom certainly does not lend itself to any quasi-static stability criterion.

The third group is concerned with the all-important oscillatory motion

For fixed controls this consists of both a rolling motion and a yawing motion. It too can be induced by any one or any combination of the possible lateral disturbances.

From the foregoing it is concluded thatunique representations in the quasi-static sense cannot be deduced for the asymmetric motion. It appears from the conclusions of reference 13 that a possibility might be a consideration of the degree of fredom about the rudder hinge line. In the investigation outlined in this report, rudder-fixed considerations were ruled out in that actual flight tests showed good agreement with predicted oscillations up to Mach numbers corresponding to a speed of 450 mph (density altitude at this speed was not specified). Insofar as a gradual deterioration in damping with speed resulting in rudder-free tests, a variation of the rudder hinge moments, both C_{h_β} and $C_{h_{\delta_r}}$, with Mach number is inferred. It is also stated in this report that this trend has been observed in high Mach number tests of other contraol surfaces. Amplification on these latter mentioned tests is not made. Two facts must be pointed out here, however. First, the above mentioned possibility is not in the quasi-static sense of Dr. Scheubel's approach. Secondly, reference 13 based its computed values at high speed on an extrapolation of a determination at an airspeed of 210 miles per hpur assuming a constant value of the stability derivatives throughout the speed range

A further complication in the establishment of quasi-static criteria for the lateral case are the inclusion or exclusion of the effects of products of inertia. These effects are discussed in detail in reference 14 and mentioned in reference 12 for a specific aircraft configuration. The effects of their inclusion was considered negligible in reference 13, but this conclusion must be made weighed in the light of the remarks made above.

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